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Algebraic Method of the Robust Stability of Interval Dynamic Systems

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Abstract

The algebraic method of researches of robust stability is considered of continuous and discrete interval dynamic systems. The results specifying and supplementing earlier known are received. The illustrating examples are given.

Keywords: robust stability, interval dynamical systems, angular polynoms of Kharitonov, polyhedron of matrixes, consecutive separate angular coefficients.

Introduction

V. L. Kharitonov's [1] work has caused huge interest in a problem of researches of a robustness of interval dynamic systems [2-12]. In the modern theory of interval dynamic systems there are two alternative approaches [10-15]: is algebraic or Kharitonov's approach; is frequency or Tsypkin's-Polyak approach.

In the algebraic or Kharitonov's direction of researches of a problem of robust stability works of many authors are widely known [2-6, 10-12, 16-19].

In works [2-6] reviews and statements of problems of robust stability which have been caused by the known of V.L.Kharitonov's work [1] are submitted.

In B. T. Polyak's work, P. S. Scherbakova [11] the concept of superstability of linear control systems is offered. At the same time superstability systems have the properties of camber allowing simple solutions of many classical tasks of the theory of control, in particular, of a problem of robust stabilization at matrix uncertainty. But essential restriction of such systems is the practical narrowness of their class, determined

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by conditions of existence of the dominating diagonal elements of a matrix of system with negative coefficients [20].

In V. M. Kuntsevich's [12] work interesting results on robust stability for linear discrete systems are received. At the same time the matrix of system is set in a class of attendants a characteristic polynom of system, i.e. in the Frobenius form [20] that also narrows a class of the considered real systems.

In B. R. Barmish works, etc. [16, 17] are offered counterexamples to Bialas [7] theorem which are cancelled in work [8].

In M. Mansour works, etc. [18, 19] are received discrete analogs of weak and strong theorems of Kharitonov's [1], which have the restrictions imposed on interval areas of coefficients or [2, 9, 13] difficult procedure of design of roots of polynoms on a piece is applied [-1, 1].

In the real work the algebraic method of research of robust stability of continuous interval dynamic systems, in case of interval matrixes of systems of a general view and without certain restrictions which foundation was initiated in works [8, 9] is considered.

Continuous Systems

1. Problem definition

Linear dynamic systems of an order of n are considered

$$\dot{x} = Ax, x(t_0) = x_0 \tag{1.1}$$

where $x = x(t) \in R^n$ is a state vector, $A \in R^{nxn}$ is an interval matrix with elements $a_{ij}, i, j = \overline{1, n},$ the representing interval sizes $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ with angular values $\underline{a}_{ij}, \overline{a}_{ij}, \underline{a}_{ij} \le \overline{a}_{ij}$.

It is required to define conditions of robust stability of system (1.1).

2. Main results

In work [8], fundamental for the considered method, results in the form of strictly proved Theorem 1 and Lemma to it about robust stability of systems (1.1) under the terms that four angular polynoms of Kharitonov's are Hurwitz ones, made on consecutive separate slopes $b_i, (\underline{b}_i, \overline{b}_i, i = \overline{1, n})$ of characteristic polynoms of system (1.1) are received:

$$f(\lambda) = \lambda^n + b_1\lambda^{n-1} + \ldots + b_n = 0 \tag{2.1}$$

We will provide these the Theorem 1 and a Lemma.

Theorem 1. *In order that position of balance x=0 of system (1.1) asymptotically was stability at all* $A \in D$*, or that the interval matrix A was stability, is necessary also sufficiently that all four angular Kharitonov's polynoms are Hurwitz ones, made on consecutive separate slopes* $b_i, (\underline{b}_i, \overline{b}_i, i = \overline{1,n})$ *of characteristic polynoms (2.1) systems (1.1).*

This theorem is provedon the basis of the following Lemma.

Lemma. *Separate slopes* $b_i, (\underline{b}_i, \overline{b}_i, i = \overline{1,n})$ *are formed as the corresponding coefficients of polynoms (1.2), or at angular values of elements* $a_{ij}, i, j = \overline{1,n},$ *of a matrix A, or at zero values of some elements (if the interval of accessory includes zero).*

As it is easy to see from a Lemma, for finding of coefficients, application of optimizing methods of nonlinear programming generally is necessary [21].

To the Theorem 1 which proof is given in the appendix of work [8] it is necessary to make the following specifying remark.

Remark. *Follows from the main argument of the proof of the Theorem 1 connected with existence of four angular Kharitonov's polynoms, that in the absence of a full set (set) of four angular polynoms of a condition of the Theorem 1 are necessary, but can be insufficient for stability of system (1.1).*

The case appropriate to the provided Remark can arise when separate slopes of polynoms (2.1) are interconnected and as a result narrow a set of angular polynoms to quantity less than four, including also multiple, coinciding polynoms.

Justice of the proved Theorem 1 is confirmed by cancellation of the known counterexamples to Bialas's theorem [7].

So, the Theorem 1 is approved on various counterexamples of the theorem of Bialas, in particular from work [16] where the matrix is considered

$$A = \Omega_r = \begin{bmatrix} -0.5-r & -12.06 & -0.06 \\ -0.25 & 0 & 1 \\ 0.25 & -4 & -1 \end{bmatrix} \qquad (2.2)$$

where $r \in [0,1]$, for which justice of the Theorem 1 is confirmed.

But in case of a matrix from [16] it is possible to consider visually justice of the Remark given above to the Theorem 1.

Really, in this case consecutive separate slopes form an incomplete set of slopes

$$b_1 = -\sum_{i=1}^{3} a_{ij} = 1.5 + r = b_2 = \sum_{i,j=1}^{3} a_{ii}a_{jj} - \sum_{i,j=1}^{3} a_{ij}a_{ji},$$
$$b_3 = \sum_{i,j,k=1}^{3} a_{ij}a_{jk}a_{kj} - \sum_{i,j,k=1}^{3} a_{ij}a_{jk}a_{ki} - a_{11}a_{22}a_{33} = 4r + 2.06,$$

as from there are separate slopes:

$$\underline{b}_1 = 1.5; \overline{b}_1 = 2.5; \underline{b}_2 = 1.5, \overline{b}_2 = 2,5; \underline{b}_3 = 2.06, \overline{b}_3 = 6.06.$$

Respectively, angular polynoms of Kharitonov in this case will be only two

$$\begin{aligned} f_1(\lambda) &= \lambda^3 + 1.5\lambda^2 + 1.5\lambda + 2.06 = f_2(\lambda), \\ f_3(\lambda) &= \lambda^3 + 2.5\lambda^2 + 2.5\lambda + 4.06 = f_4(\lambda), \end{aligned} \tag{2.3}$$

i.e. the totality of 4 angular polynoms specified in work [8] won't be.

Therefore, on angular polynoms (2.3) the system (1.1) everywhere at $r \in [0,1]$ will be stability, though it is known that at $r \in [0.5 - \sqrt{0.06},\ 0.5 + \sqrt{0.06}]$ this system it is unstability.

The Theorem 1 and a Lemma allow to solve a problem about a costal hypothesis for polyhedrons of matrixes [2].

It is known [2] what a polyhedron of matrixes is called sets

$$P = \left\{ P_s = \sum_{i=1}^{m} s_i P_i : s_i \ge 0, i = \overline{1,m}, \sum_{i=1}^{m} s_i = 1 \right\} \tag{2.4}$$

where P_i, $i = \overline{1,m}$ - constant matrixes.

In work [2] the hypothesis of stability conditions of a polyhedron P (2.4) in the following look is formulated:

Hypothesis. *The polyhedron P is stability in only case when edges P are stability, i.e. matrixes* $sP_i + (1-s)P_j$*, it is stability at any* $i, j = \overline{1,m}, s \in [0,1]$.

But in work [17] on counterexamples it is shown, that this hypothesis is incorrect for strictly

Hurwitz's a case.

Contradictions in a costal hypothesis are resolved on the basis of the following costal Theorem 2, proved in work [8].

Theorem 2. *Stability of a polyhedron of matrixes P necessary also sufficiently that convex edges P were stability, i.e. matrixes* $s_1P_i + s_2P_j$ *, it is stability at any* $i, j = \overline{1,m}, s_1 \in [-1.0], s_2 \in [0,1]$.

In this case the polyhedron of matrixes P is presented in the form:

$$P = \left\{ P_s = P_{s1} + P_{s2} : P_{s1} = \sum_{i=1}^{m} s_{1i}P_i, \quad P_{s2} = \sum_{i=1}^{m} s_{2i}P_i : s_{1i} + s_{2i} = s_i \geq 0, i = \overline{1,m}, \sum_{1}^{m} s_i = 1 \right\}.$$

Justice of the Theorem 2 is also confirmed by cancellation of all known counterexamples from work [17].

3. Examples

We will review examples which visually explain sense of the Remark to the Theorem 1.

Example 1. We will show that the system (1.1) with an interval matrix of a look $A = \Omega_r$ (2.2), which doesn't possess a totality from four characteristic Kharitonov's polynoms is an uncertain robustly.

Really, if we put r_1 = 0,5-√0,06, r_2 = 0,5 + √0,06, then as is well-known [16] system (1.1) are unstability at $r_1 < r < r_2$, and in r intervals r϶ [0, r_1) and r ϵ (r_2, 1] this system is stability. If now to assume that coefficients of a characteristic polynom (2.1) don't depend from each other and we have four angular polynoms of system (1.1), then it is easy to calculate that only in small parts of the intervals of robust stability of r stated above r϶ [0, r_1) and r ϵ (r_2, 1] takes place of stability of all four characteristic polynoms, namely at r ϵ [0, 0,475) and r ϵ (0,9617, 1].

So for example, for *r* interval $r \in [0, r_1)$ we have the following four angular polynoms:

$$f_1(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + 1{,}5\,\lambda + (2{,}06 + 4r_1),$$
$$f_2(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + (1{,}5 + r_1)\,\lambda + (2{,}06 + 4r_1),$$
$$f_3(\lambda) = \lambda^3 + (1{,}5 + r_1)\,\lambda^2 + 1{,}5\,\lambda + 2{,}06,$$
$$f_4(\lambda) = \lambda^3 + (1{,}5 + r_1)\,\lambda^2 + (1{,}5 + r_1)\,\lambda + 2{,}06,$$

from which two first are unstability, and the following two is stability.

This example shows that because of dependence between coefficients of a characteristic polynom (2.1), here incomplete set of two angular polynoms really takes place, and we can't unambiguously establish intervals of robust stability of system (1.1), as well as followed according to the Remark to the Theorem 1.

Example 2. Let the characteristic polynom of interval system (1.1) in a look be set

$$f(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + 2{,}5\,\lambda + b_3, \tag{3.1}$$

where coefficient $b_3 \in [2, 3]$. Such case of system (1.1) is possible for example, at the Frobenius or accompanying form of a matrix A.

Then, four angular characteristic polynoms of system (1.1) will be the following:

$$f_1(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + 2{,}5\,\lambda + 3,$$
$$f_2(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + 2{,}5\,\lambda + 3,$$
$$f_3(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + 2{,}5\,\lambda + 2,$$
$$f_4(\lambda) = \lambda^3 + 1{,}5\,\lambda^2 + 2{,}5\,\lambda + 2.$$

As it is easy to see, in this case all four characteristic polynoms of system (1.1) are stability ($b_1 b_2 > b_3$) and the system (1.1) is robust stability. If, we put $b_3 \in [2{,}06,\ 4{,}06]$, then in this case, the first two polynoms of $f_1(\lambda)$ and $f_2(\lambda)$ are unstability, and two the following $f_3(\lambda)$ and $f_4(\lambda)$ are stability, and consequently the system (1.1) is robust unstability.

In this example we have considered two cases of interval system (1.1) when there are two couples of coinciding or multiple characteristic polynoms, but at the same time there are full sets of four Kharitonov's polynoms, unlike a case of system (1.1) with a matrix $A = \Omega_r$ (2.2), where there is no totality from four angular characteristic polynoms, but only two angular polynoms, owing to tough dependence of coefficients of b_i, i = 1, 2, 3 on parameter r. Therefore, in both cases of the reviewed Example 2 according to the Remark to the Theorem 1 it is possible to draw quite certain conclusion on robust stability or instability of interval system (1.1) with a characteristic polynom (3.1), while in a case with a type matrix $A = \Omega_r$ we can't make a certain conclusion about robust stability (here it should be noted, that in work [8] the conclusion on this case is drawn illegally).

Discrete Systems

4. Problem definition

Discrete linear dynamic systems of an order of n are considered

$$x\,(m+1) = Ax(m),\ m = 1, 2, 3, \ldots, \qquad (4.1)$$

where, $x(m)$ is a state vector, $A \in R^{nxn}$ is an interval matrix with elements $a_{ij}, i, j = \overline{1, n}$, the representing interval sizes $a_{ij} \in [\underline{a}_{ij}, \overline{a}_{ij}]$ with angular values $\underline{a}_{ij}, \overline{a}_{ij}, \underline{a}_{ij} \le \overline{a}_{ij}$.

It is required to define conditions of robust stability of systems (4.1).

5. Main Results

It is known, that the publication of work [1] has given an impulse for search by many reseachers of discrete analogs of theorems of Kharitonov [2-7, 9]. So in work [2] it is specified that "the discrete option of a Kharitonov’s condition of four polynomials is absent". But here it is noted that now [19, 20] discrete analogs weak are received and strong theorems of Kharitonov. But these analogs of theorems of Kharitonov have the certain restrictions imposed on interval areas of coefficients [2]. These restrictions have been removed in works [9, 13] where analogs of theorems of Kharitonov with use received of the theorem of Schur. Also in [9, 13] the theorems which are discrete analogs of results of work [8] on interval matrixes and polyhedrons of matrixes are formulated.

Further, generalization of the results received in work [9,13] taking into account the conclusions given above for continuous systems is considered.

For discrete systems, using z - transformation, obtained interval characteristic polynom

$$f(z) = \det\,(zI - A) = \sum_{i=0}^{n} b_i z^{n-i}\ ,\ b_i \in [\,\underline{b}_i\,, \overline{b}_i\,],\ \ \underline{b}_i\ \le\ \overline{b}_i\,. \qquad (5.1)$$

For definition of stability conditions we will use Schur theorem, i.e. look conditions $|b_{\mathrm{o}}| > |b_n|$, for the sequence of the polynoms determined by recurrence relations

$$f_i\,(z) = [b_{\mathrm{o}}f(z) - b_nf(1/z)z^n]/z, \ldots, f_{i+1}(z) = [b_{\mathrm{o},i}f_i(z) - b_{n,i}f_i(1/z)z^{n-1}]/z, \qquad (5.2)$$

where $b_{o,i}$, $b_{n,i}$ are respectively the senior and younger coefficients of polynom $f_i(z)$, $i = 1, 2, 3, \ldots, n$-2.

Definition

We will call *change points* for coefficients b_i, $i = 0, 1, 2, \ldots, n$, points on the valid axis in which there are transitions of roots of a polynom (5.1), through a which single circle on the plane of roots, and *change* *intervals* are respectively intervals in roots are or inside, or out of a single circle.

In work [9] the main results on definition of conditions of robust stability of discrete interval systems in the form of the corresponding Theorems 1-6 are formulated. At the same time it should be noted that as it is stated above on page 3, for a case of continuous systems [8], justice of the Theorem 5 has the restriction caused by the Remark to the Theorem of 1 work [8] i.e. the Theorem 5 is right at a totality from 4 various polynoms of Kharitonov.

Justice of results [9, 13] concerning an analog of the strong theorem of Kharitonov are shown on the known counterexamples from [2], etc.

Thus,the algorithm of definition of robust stability discrete interval dynamic systems will be the following.

1. Using lemma formulas to the theorem 1 [8], optimization on elements $a_{ij} \in \left[\underline{a}_{ij}, \overline{a}_{ij},\right], i, j = \overline{1, n}$ of interval matrix A, there are separate coefficients $b_i \in \left[\underline{b}_i, \overline{b}_i\right], i = \overline{0, n}$, of interval characteristic polynom (5.1).

2. Four polynoms of Kharitonov corresponding to an interval polynom (5.1) are defined

$f_1(z):\left\{\underline{b}_0, \underline{b}_1, \overline{b}_2, \overline{b}_3, \underline{b}_4 ...,\right\}$; $f_2(z):\{\underline{b}_0, \overline{b}_1, \overline{b}_2, \underline{b}_3, \underline{b}_4 ...,\}$;

$f_3(z):\left\{\overline{b}_0, \underline{b}_1, \underline{b}_2, \overline{b}_3, \underline{b}_4 ...,\right\}$; $f_4(z):\{\overline{b}_0, \overline{b}_1, \underline{b}_2, \underline{b}_3, \overline{b}_4 ...,\}$;

3. N of inequalities of a look (Item 2) specified in the Appendix of work [9] are formed.

4. Concerning each coefficient $b_i, i = \overline{0, n}$, including other coefficients fixed, consistently there are *change points* for all four polynoms of Kharitonov and on all n to inequalities (see item 3), since smaller orders.

5. If all *change points* on all coefficients $b_i, i = \overline{0, n}$, not belong reaped to the set intervals, then the initial polynom (system) is stability, otherwise is unstability.

Conclusion

The algebraic method of researches of robust stability of interval dynamic systems considered in this work is further development of the main results of works [8, 9], which allows to solve a problem of robust stability at a general view of an interval matrix of system. At the same time the method is directed for the solution of problems of robust stability both for linear continuous dynamic systems, and for linear discrete dynamic systems.

It should be noted, that the Remark to the Theorem 1 essentially specifies results of work [8], namely emphasizes necessary of a totality from four angular polynoms of Kharitonov (taking into account frequency rate of polynoms) for definition of robust stability of interval dynamic systems. Also, conditions of necessary and sufficiency on the Theorem 1 correspond to the angular separate coefficients, defined consistently from 1st to n coefficient of a characteristic polynom of system which can be found with use of methods of nonlinear programming [21].

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