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A Robust Derivation of the Black-Scholes Partial Differential Equation System without the Self-Financing Hypothesis

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Abstract

Delta hedging is the core of the derivation of the well-known Black-Scholes formula for the price of European options. When Ito calculus is used faithfully without the self-financing hypothesis, the dependence of the delta on both time and the underlying asset price variables is induced and subsequently a consistent version of the Black-Scholes partial differential equation system is derived for the option price and the delta. This paper proposes the system as a possible starting point of a more robust study of mathematical option pricing.

Keywords: Black-Scholes partial differential equation, Ito calculus, Self-financing, No arbitrage, Option pricing.

1. Introduction

The Black-Scholes formula obtained by the seminal paper, "The pricing of options and corporate liabilities", published in the Journal of Political Economy [1] is a well-known mathematical tool providing a theoretical estimate of the price of European options in finance. This scientific result led to a boost in option trading around the world as noted by [2]. It is popularly used by option market practitioners even if there is a need for it to be corrected due to well-known discrepancies such as "volatility smiles", i.e., implied volatility patterns that arise in pricing options, which began to be observed, particularly after the crash called "Black-Monday" of 1987 [3].

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A core of the derivation of the Black-Scholes formula is to eliminate the risk of an option by buying or selling an appropriate amount, called the delta, of the underlying asset in the right way. This type of risk elimination is called delta hedging and it became the basis of more complicated hedging strategies used by market participants these days. When the formula is derived (cf. [4] for example), the delta is assumed to be constant during the infinitesimal time step. This 'intuitive' assumption is related to the so-called self-financing construction of a portfolio; there is no exogenous infusion or withdrawal of money so that the purchase of a new asset must be financed by the sale of an old one.

There is no doubt that a dominant proportion of mathematical research on option pricing and hedging constructs self-financing trading strategies satisfying some conditions including the no-arbitrage condition. Frictions in markets and shortfall risks and etc., however, became important subjects to be dealt with as discussed by [5]. Further, the study of non-self-financing trading strategies in this type of economic environments can be found in [6], [7] and [8] in the context of option pricing with consumption.

We consider directly the original derivation of the Black-Scholes partial differential equation without the self-financing condition. We use the Ito product rule to derive a robust version of the Black-Scholes partial differential equation for European options. It turns out that it is a coupled system of partial differential equations for the option price and delta.

In the next section, we provide the details of the derivation by using the Ito calculus (cf. [9] for example). Then, in Section 3, we prove that delta has to be a function of both time and the underlying price. Section 4 concludes.

2. A Partial Differential Equation System for European Option

Let the price of an underlying asset follow a geometric Brownian Motion given by the stochastic differential equation

$$dS = \mu S dt + \sigma S dW, \tag{1}$$

where μ (mean return rate) and σ (volatility) are constants and W is a standard Brownian motion.

Given the underlying asset price S at time t, let $V(t,S)$ be the European option price. Then, from the Itô formula, the differential of V is

$$dV = \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt + \sigma S \frac{\partial V}{\partial S} dW. \quad (2)$$

Now, we consider a (non-trivial) portfolio $(1, -\Delta)$ consisting of one option and $-\Delta$ number of underlying asset. Its value, denoted by Π, is given by

$$\Pi(t,S) = V(t,S) - \Delta(t,S)S. \quad (3)$$

In this paper, Δ is considered to be a function of time t and the underlying asset price S and it is to be determined later. If we follow the Ito product rule faithfully, the differential $d\Pi$ is given by

$$d\Pi = dV - (\Delta dS + S d\Delta + d\Delta dS). \quad (4)$$

So, the self-financing condition is not required any more. That condition would result in $d\Pi = dV - \Delta dS$ as assumed in the traditional Black-Scholes theory. Since

$$d\Delta = \left(\mu S \frac{\partial \Delta}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Delta}{\partial S^2} + \frac{\partial \Delta}{\partial t} \right) dt + \sigma S \frac{\partial \Delta}{\partial S} dW$$

holds from the Ito formula, we have $d\Delta dS = (\sigma S)^2 \frac{\partial \Delta}{\partial S} dt$. Then from (1), (2) and (4) we have

$$d\Pi = \left[\left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) - \Delta \mu S - S \left(\frac{\partial \Delta}{\partial t} + \mu S \frac{\partial \Delta}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Delta}{\partial S^2} \right) - \sigma^2 S^2 \frac{\partial \Delta}{\partial S} \right] dt$$

$$+ \sigma S \left(\frac{\partial V}{\partial S} - \Delta - S \frac{\partial \Delta}{\partial S} \right) dW. \quad (5)$$

If Δ is chosen such that

$$\Delta + S \frac{\partial \Delta}{\partial S} = \frac{\partial V}{\partial S} \quad (6)$$

is satisfied, then the portfolio becomes effectively risk-free, i.e., Π is totally deterministic. So, if there is no arbitrage in the market, the return rate of the portfolio must be the risk-free interest rate r so that

$$d\Pi = r\Pi dt = r(V - \Delta S) dt \quad (7)$$

holds under the assumption of no transaction costs. Therefore, from (5), (6) and (7), we obtain a partial differential equation given by

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r(S\frac{\partial V}{\partial S}-V)=\delta(t,S), \tag{8}$$

where δ is a function (of underlying asset price and time) given by

$$\delta(t,S)=S\left(\frac{\partial\Delta}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2\Delta}{\partial S^2}+(r+\sigma^2)S\frac{\partial\Delta}{\partial S}\right). \tag{9}$$

The required boundary conditions of the partial differential equation system (6) and (8) are

$$V(t,0)=0,\quad \Delta(t,0)=0,$$

$$V(t,S)\sim S \quad \text{as} \quad S\to\infty, \tag{10}$$

$$V(T,S)=h(S),$$

where the payoff function h is given by $h(S)=(S-K)^+$ for European call options with strike price K. Similar alternative conditions apply for the European put option.

The results obtained above are summarized as a theorem as follows.

Theorem 1. Under the no-arbitrage assumption but without the self-financing condition, the option price V and the delta Δ satisfy a coupled partial differential equation system given by

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r(S\frac{\partial V}{\partial S}-V)=\delta(t,S), \tag{11}$$

$$\Delta+S\frac{\partial\Delta}{\partial S}=\frac{\partial V}{\partial S} \tag{12}$$

under the boundary conditions (10), where δ is a function given by (9).

Note that the equation for V is parabolic while the one for Δ is hyperbolic.

3. Delta

In this section, we prove that the hedge ratio of the European option has to depend on both time and the underlying asset price in order to maintain consistency in the system.

Theorem 2. If the option price V and the delta Δ satisfy the system given by (11) and (12), then Δ (the number of underlying asset for hedging) has to depend on both of the variables t and S in order to maintain consistency in the system.

Proof. Suppose that Δ does not depend on at least one of the variables t and S.

First, if Δ is assumed to be independent of t, then δ in (9) becomes

$$\delta(t,S) = \frac{1}{2}\sigma^2 S^3 \frac{\partial^2 \Delta}{\partial S^2} + (r+\sigma^2)S^2 \frac{\partial \Delta}{\partial S}$$

and thus (11) and (12) remain still as a coupled system of equations. However, from (12), the left-hand side $\Delta + S\frac{\partial \Delta}{\partial S}$ of it is independent of t, whereas the right-hand side $\frac{\partial V}{\partial S}$ of it depends on t, violating consistency.

Next, if Δ is assumed to be independent of S, then

$$\delta(t,S) = S\frac{\partial \Delta}{\partial t}$$

and thus (11) and (12) become decoupled as follows.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r(S\frac{\partial V}{\partial S} - V) - S\frac{\partial^2 V}{\partial t \partial S} = 0,$$
$$\Delta = \frac{\partial V}{\partial S}.$$

The equation for V is hyperbolic and so has wave-like solutions. However, it also creates a consistency problem since $\Delta = \frac{\partial V}{\partial S}$ depends on S. □

If Δ is assumed to be independent of both t and S, then, from (9), $\delta(t,S) = 0$ and thus (11) and (12) are reduced to

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+r(S\frac{\partial V}{\partial S}-V)=0, \tag{13}$$

$$\Delta=\frac{\partial V}{\partial S}, \tag{14}$$

which corresponds to the well-known traditional Black-Scholes case. In this case, the following formula for V (call) is well-known.

$$V(t,S)=S\Phi(d_1)-Ke^{-r(T-t)}\Phi(d_2),$$

where Φ is the standard normal cumulative distribution function and

$$d_{1,2}(t,S)=\frac{\ln(S/K)+(r\pm\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Obviously, the Black-Scholes option price V depends on t and is not linear in S and so $\Delta=\frac{\partial V}{\partial S}$ is not consistent with the assumption of its independence of any of the variables t and S.

The argument above has been done so far without the self-financing condition. If the self-financing hypothesis is applied to (4) (even though Δ depends on the variables t and S), then (4) becomes

$$d\Pi=dV-\Delta dS.$$

Based on this identity, the same argument as above would lead to system (13) and (14) as done in the original Black-Scholes theory.

4. Conclusion

This paper obtains a coupled system given by (11) and (12) for the option price and delta without the self-financing hypothesis. It should be a possible starting point of a more robust study for the Black-Scholes theory of option pricing in mathematical finance when the self-financing condition is too restrictive. It is a quite complicated system of equations to be solved. Note that using a classical rule such as $\Delta+S\frac{\partial\Delta}{\partial S}=\frac{\partial\Delta}{\partial S}(S\Delta)$ would not apply under the rules of Ito Calculus for the simplification of the above system. As a first step toward the further development of the proposed study, it remains to solve the system analytically, asymptotically or numerically, and then to test it against time series data for $V(t,S)$.

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