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Counter Examples for Lemma 3.18 Of [1]

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Abstract

In proof of Lemma 3.18 in [1] acclaim that, at least one of the eigenvalues of Grammian matrix G_{11} has distance $\frac{\eta}{(n-1)}$ from $\{0,1\}$. In this paper, we show by two counter examples that this claim is incorrect and we say state correct it with proof it.

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1. Introduction

In this paper we let, H is a seprable Hilbert space with dimension d, and n be a countable index set that $(n,d) = 1$.

Definition 1.1. [2] A family of vectors $F = \{f_i\}j \in J$ is a frame for a Hilbert space H if there are constants $0 < A \leq B < \infty$ such that for all $f \in H$

$$A\|f\|^2 \leq \sum_{j\in J} \left|< f, f_j >\right|^2 \leq B\,\|f\|^2 \,, \forall f \in H\,.$$

We call A and B the frame bounds. If we can choose $A = B$ then F is a A-tight frame and if $A = B = 1$, it is a parseval frame. If all the frame vectors have the same norm, it is an equal-norm frame. We call A and B the frame bounds. If $F = \{f_i\}j \in J$ possesses an upper frame bound,but not necessarily a lower bound, we call it is a Bessel sequence with Bessel bound A. The analysis operator of the frame is the map

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$$V: H \rightarrow l^2(J)$$

$$V(f) := \{< f, f_j >\}_{j \in J} .$$

Its adjoint is the synthesis operator which maps $a \in l^2(J)$

$$V^*: l^2(J) \rightarrow H$$

$$V^*(f) := \sum_{j \in J} a_j f_j .$$

The frame operator is the positive, self-adjoint invertible operator

$$S: H \rightarrow H, \quad Sf = V^*Vf = \sum_{j \in J} < f, f_j > f_j .$$

The Grammian is the matrix G with entries $G_{j,k} = \langle f_j, f_k \rangle$ so that $G_{j,k} = (VV^*)_{k,j}, k, j \in \{1, \ldots, n\}$.

Definition 1.2. A frame $\{f_j\}_{j=1}^{n}$ for a d-dimensional real or complex Hilbert space H is $\epsilon-$ nearly equal-norm with constant c if

$$(1-\epsilon)c \leq \|f_j\| \leq (1+\epsilon)c , \ \ \forall j \in \{1,2, \ldots, n\}.$$

Definition 1.3. [1] Let n, $d \in N$ be relatively prime

$$\eta = min_{\substack{n_1 < n \\ d_1 < d}} \left|\frac{d}{n} - \frac{d_1}{n_1}\right| . \tag{1.1}$$

Example 1.4. Let $d = 3, n = 5$ then

$$d_1 = \{1,2\}, \ \ n_1 = \{1,2,3,4\},$$

$$\frac{d_1}{n_1} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 2, \frac{2}{3}\right\},$$

$$\eta = \min\left\{\left|\frac{d}{n} - \frac{d_1}{n_1}\right|\right\} = min \ \left\{\left|\frac{3}{5} - \frac{d_1}{n_1}\right|\right\} = \min\{\frac{2}{5}, \frac{1}{10}, \frac{4}{1}, \frac{7}{3}, \frac{7}{6}\} = \frac{1}{3} \ .$$

2. Counter Examples

At least one of the eigenvalues of Grammian matrix G_{11} has distance $\frac{\eta}{n-1}$ from {0,1}, by the definition of distance between a point and setis equivalent to this acclaim, at least for one eigenvalue (λ) of eigenvalues related Grammian matrix $G_{11}, \lambda = \frac{\eta}{n-1}$ or $1 - \lambda = \frac{\eta}{n-1}$.

Initial Example Counter 2. 1. We let dim *H* = 2 and number ofvector frame *n* = 3. Let frame $F = \{F_1, F_2, F_3\}$ in Hilbert space *H*, that

$$F_1 = \left(\sqrt{\frac{2}{3}}\,,0\right),\quad F_2 = \left(\frac{-1}{\sqrt{6}}\,,\frac{1}{\sqrt{2}}\right),\quad F_3 = \left(\frac{-1}{\sqrt{6}}\,,\frac{-1}{\sqrt{2}}\right)$$

We first show that *F* is Parseval frame, Since for arbitrary vector $F = \{F_1, F_2\}$ in *H*,

$$|< f,\ F_1 >|^2 = \frac{2}{3}F_1^2\ ,\ |< f,\ F_2 >|^2 = \frac{1}{6}f_1^2 + \frac{1}{2}f_2^2 - \frac{1}{\sqrt{2}.\sqrt{6}}f_1f_2,$$

and

$$|< f,\ F_3 >|^2 = \frac{1}{6}f_1^2 + \frac{1}{2}f_2^2 + \frac{1}{\sqrt{2}.\sqrt{6}}f_1f_2,$$

thus

$$\sum_{n=1}^{3}|< f, \qquad F_n >|^2 = \ f_1^2 +\ f_2^2 = \|f\|^2.$$

Thus F is a Parseval frame.

We procure for *n* = 3 and *d* = 2, η at method under

$$d_1 = \{1\}, \qquad\qquad N_1 = \{1,2\},$$

thus

$$\eta = \min\left\{\frac{1}{3},\frac{1}{6}\right\} = \frac{1}{6}\ ,$$

Therefore $\frac{\eta}{n-1} = \frac{1}{12}$.

Then constitute the Grammian matrix *G* relative to Parseval frame *F*,

$$G = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{-1}{3} & \frac{2}{3} \end{bmatrix}$$

And we form the corresponding blocks in the Grammian so that, G_{11} is a Hermitian matrix and G_{12}^*=G_{21}

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

We only can consider two states for block G_{11}

(1) $G_{11} = [\frac{2}{3}]$ is block 1×1 , thus $\lambda(G_{11}) = \frac{2}{3}$, it is clear that $1 - \frac{2}{3} = \frac{1}{3}$ and $\frac{2}{3}$ are apposite to $\frac{1}{12}$.

(2) G_{11} is block 2×2 at form under

$$G_{11} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{bmatrix},$$

It is clear that, eigenvalues of G_{11} are equal to 1, $\frac{1}{3}$. If $\lambda(G_{11}) = 1$ it is clear that 1, 1-1 are apposite to $\frac{1}{12}$, if $\lambda(G_{11}) = \frac{1}{3}$ it is clear that $1 - \frac{1}{3} = \frac{2}{3}$ and $\frac{1}{3}$ are opposite to $\frac{1}{12}$.

Thus in two above states, we observed that none of eigenvalues relative to G_{11} not have distance $\frac{\eta}{n-1} = \frac{1}{12}$ of set $\{0, 1\}$.

Initial Example counter 2. 2. We let dim *H* = 3 and number of vector frame *n* = 4. Let frame $F = \{F_1, F_2, F_3, F_4\}$ in Hilbert space *H*, that

$$F_1 = \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad F_2 = \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right),$$

$$F_3 = \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right), \quad F_4 = \left(\frac{-1}{2}, \frac{-1}{2}, \frac{-1}{2}\right)$$

It is clear that F is a Parseval frame for Hilbert space *H*.

We procure for *n*=4 and *d*=3, $\eta = \frac{1}{12}$ and $\frac{\eta}{n-1} = \frac{1}{36}$. Then constitute the Grammian matrix *G* relative to Parseval frame *F*,

$$G = \begin{bmatrix} \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix}$$

Similar to before, we form the corresponding blocks in the Grammian *G* so that G_{11} is a Hermitian matrix, we only can consider three states for block G_{11}

(1) $G_{11} = [\frac{3}{4}]$ is block 1×1 thus $\lambda(G_{11}) = \frac{3}{4}$. It is clear that $\frac{3}{4}$ and $1 - \frac{3}{4} = \frac{1}{4}$ are opposite to $\frac{1}{36}$.

(2) G_{11} is block 2×2 at form under

$$G_{11} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} \end{bmatrix},$$

It is clear that, eigenvalues of G_{11} are equal to 1, $\frac{1}{2}$. If $\lambda(G_{11}) = 1$ it is clear that 1, 1-1 are opposite to $\frac{1}{36}$. If $\lambda(G_{11}) = \frac{1}{2}$ it is clear that $1 - \frac{1}{2} = \frac{1}{2}$ and $\frac{1}{2}$ are opposite to $\frac{1}{36}$.

(3) G_{11} is block 3×3 at form under

$$G_{11} = \begin{bmatrix} \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{3}{4} & \frac{-1}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{3}{4} \end{bmatrix},$$

It is clear that, eigenvalues of G_{11} are equal to $\lambda_1 = \lambda_2 = 1, \lambda_3 = \frac{1}{4}$. If $\lambda(G_{11}) = 1$ it is clear that 1, 1-1 are opposite to $\frac{\eta}{n-1} = \frac{1}{36}$. If $\lambda(G_{11}) = \frac{1}{4}$ it is clear that $1 - \frac{1}{4} = \frac{3}{4}$ and $\frac{1}{4}$ are opposite to $\frac{\eta}{n-1} = \frac{1}{36}$.

Thus in three above states, we observed that none of eigenvalues relative to G_{11} not have distance $\frac{\eta}{n-1} = \frac{1}{36}$ of set $\{0,1\}$.

3. Correction Proof Of Lemma 3. 18

In this section, we show by the using of information Lemma 3.18 in [1] that, at least one of the eigenvalues of Grammian matrix G_{11} has distance $\frac{\eta}{(n-1)}$ from $\{0,1\}$ or equivalently, at least for one eigenvalue (λ) of eigenvalues related Grammian matrix $G_{11}, \lambda > \frac{\eta}{n-1}$ or $1 - \lambda > \frac{\eta}{n-1}$.

Lemma 3.1. Let $n \geq 2, \eta$ as defined above, and let $F = \{f_j\}_{j=1}^{n}$ be a Parseval frame for a d-dimensional Hilbert space, then the variance of the random variable $W: C \mapsto W(F^{(C)})$ on the torus T^n equipped with the uniform probability measure σ_n is bounded below by

$$\frac{16\eta}{(n-1)^7}(U(F))^2 \leq \int_{T^n} (W(F^{(C)}))^2 d\sigma_n.$$

Corollary 3.2. Let $n \geq 2, \eta$ as defined above, and let $F = \{f_j\}_{j=1}^{n}$ be a Parseval frame for a d-dimensional Hilbert space, if G be Grammian matrix related to F that blocked at form

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Then at least one of the eigenvalues of Grammian matrix G_{11} has distance $\frac{\eta}{(n-1)}$ from $\{0,1\}$ or equivalently, at least for one eigenvalue (λ) of eigenvalues related Grammian matrix $G_{11}, \lambda > \frac{\eta}{n-1}$ or $1 - \lambda > \frac{\eta}{n-1}$.

Proof. Since Gis aGrammian matrix therefore G is a positive definite matrix [3], thus $G_{11} \geq 0$. Since G is a Grammian matrix therefore G is a Hermitian matrix thus G_{11} is a Hermitian matrix. We now arrange eigenvalues of G and G_{11}

$$0 \leq \lambda_{min}(G) \leq \lambda_{min}(G_{11}) \leq \cdots \leq \lambda_{max}(G_{11}) \leq \lambda_{max}(G),$$

and since G_{11} is a Hermitian matrix, thus

$$0 \leq \lambda_{min}(G_{11}) x^* x \leq x^* G_{11} x \leq \lambda_{max}(G_{11}) x^* x \leq 1 x^* x ,$$

therefore

$$x^* I\, x - x^* G_{11}\, x \geq 0$$

Thus $G_{11} \leq I$ and conseqently $\lambda(G_{11}) \in [0, 1]$. If $(n, d) = 1$ and the vectors are sufficiently near equal- norm, then the diagonal entries of G_{11} are close to $\frac{d}{n}$ and summing them does not give an integer. Therefore, not all eigenvalues are 0 or 1 or equivalently there is at least one eigenvalue (λ) related to G_{11} in (0, 1) .Since in proof of lemma 3. 18 in [1] argue on function $\lambda(1-\lambda)$ and the function $\lambda \longrightarrow \lambda(1-\lambda)$ is bounded below by $\lambda \longrightarrow \frac{\lambda}{2}$ on $[0, \frac{1}{2}]$ and by $\lambda \longrightarrow \frac{1}{2} - \frac{\lambda}{2}$ on $[\frac{1}{2}, 1]$. Thus without lessen of totality, we let $\lambda \in [0, \frac{1}{2}]$. Since $n \geq 2$ it is clear that $\frac{\eta}{n-1} < \frac{1}{2}$. Now consider 2 state for $\lambda \; and \; \frac{\eta}{n-1}$,

$$\lambda > \frac{\eta}{n-1} \tag{3.1}$$

and

$$\lambda < \frac{\eta}{n-1} \Longrightarrow 1-\lambda > \frac{\eta}{n-1} \tag{3.2}$$

By relation (3.1) and (3. 2), we gives

$$\min\{\lambda, 1-\lambda\} > \frac{\eta}{n-1} \ .$$

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