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On q-Baskakov-Durrmeyer-Stancu Operators in Approximation Theory

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Abstract

This paper is the extension of Aral-Gupta [1] by the use of Stancu type generalization of q-Baskakov-Durrmeyer operators. We establish some important relations for these operators which provide an approximation process in the polynomial weighted space of continuous functions on $[0,\infty)$. The rate of convergence and weighted approximation properties are also obtained.

Keywords: q-Baskakov-Durrmeyer operators, Stancu type generalization, Rate of Convergence, Modulus of continuity, Weighted approximation.

1. Introduction

In the approximation theory, q-calculus makes our research very interesting. In the year 1987, first q-analogue of classical Bernstein polynomials was given by A. Lupas [4]. The most important q-analogue of the Bernstein polynomials was introduced by Phillips [8] in 1997. After that many researchers worked in this direction and proposed many types of q-operators and motivated their various properties related to special functions, number theory and convergence behaviour. Gupta et al. [5] established the generating functions of some q-basis functions. In approximation theory, the convergence is very important. Therefore, in this context we mention some of the results for convergence of q-discrete operators due to [1], [3] etc.

Discrete operators are not possible to approximate the integrable functions. V. Gupta [3] introduced

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an important q-analogue of the Bernstein Durrmeyer operators based on q-Beta function of first kind in 2008. Later, in 2010, based on q-Beta function of second kind, Aral and Gupta [1] introduced q-Baskakov-Durrmeyer operators.

The main purpose of this paper is to obtain a local approximation theorem and a rate of convergence of the new operators as well as their weighted approximation properties. The processes turn out to have a better order of approximation in a certain subspace of continuous functions. Therefore for $f \in C[0,\infty)$, $q > 0$ and $n \in N$, Aral-Gupta introduced q-Baskakov operators such as

$$\begin{aligned} B_{n,q}(f;x) &= \sum_{v=0}^{\infty} \begin{bmatrix} n+v-1 \\ v \end{bmatrix} q^{\frac{v(v-1)}{2}} \frac{x^v}{(1+x)_q^{n+v}} f\left(\frac{[v]_q}{q^{v-1}[n]_q}\right) \\ &= \sum_{v=0}^{\infty} p_{n,v}^q(x) f\left(\frac{[v]_q}{q^{v-1}[n]_q}\right). \end{aligned} \tag{1}$$

Next to it, by taking $q \in (0,1)$, they constructed the linear positive operators

$$D_n^q(f,x) = [n-1]_q \sum_{v=0}^{\infty} p_{n,v}^q(x) \int_0^{\infty/A} p_{n,v}^q(t) f(t) d_q t, \tag{2}$$

where

$$p_{n,v}^q(x) = \begin{bmatrix} n+v-1 \\ v \end{bmatrix}_q q^{v^2/2} \frac{x^v}{(1+x)_q^{n+v}}, \quad x \in [0,\infty) \tag{3}$$

for every real valued continuous and bounded function f on $[0,\infty)$. Also it can be observed that in case $q = 1$, the above operators reduce to the original Baskakov-Durrmeyer operators discussed by Sahai et al. [9], P. Maheshwari [6].

Motivated by the recent studies, now we propose the Stancu type generalization [10] of the q-Baskakov-Durrmeyer operators. Actually the Stancu variant is based on two parameters α and β satisfying $0 \le \alpha \le \beta$. It generalizes the original operators. So for $0 < q < 1$ and $x \in [0,\infty)$, we propose q-Baskakov-Durrmeyer-Stancu operators

$$D_{n,\alpha,\beta}^{q}(f;x)=[n-1]_q\sum_{v=0}^{\infty}p_{n,v}^{q}(x)\int_0^{\infty/A}p_{n,v}^{q}(t)f\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right)d_q t, \qquad (4)$$

for every $f\in[0,\infty)$ and $p_{n,v}^{q}(t)$ defined in (3). If $q=1,$ the above operators reduce to origional Baskakov-Durrmeyer-Stancu operators of which some properties are discussed by Maheshwari-Sharma [7]. Obviously for $\alpha=\beta=0$ operators (4) reduce to q-Baskakov-Durrmeyer operators (2).

Before starting our work, it is necessary to recall the concepts of q-calculus, which can be studied in the book written by Aral et al. [2].

2. Moment Estimation and Auxiliary Results

In this section, we estimate certain basic results such as moments and some important lemmas.

Lemma 1. [1] For $B_{n,q}(t^m;x), m=0,1,2,$ we have

$$B_{n,q}(1;x)=1;$$
$$B_{n,q}(t;x)=x;$$
$$B_{n,q}(t^2;x)=x^2+\frac{x}{[n]_q}\left(1+\frac{x}{q}\right)$$

Lemma 2. [1] For $n\in N$ and $q\in(0,1),$ we have

$$D_n^q(1;x)=1,\quad n\geq 1;$$
$$D_n^q(t;x)=\left(1+\frac{[2]_q}{q^2[n-2]_q}\right)x+\frac{1}{q[n-2]_q}=\frac{[n]_q x+q}{q^2[n-2]_q},\quad n>2;$$
$$D_n^q(t^2;x)=\left(1+\frac{[3]_q}{q^3[n-3]_q}+\frac{[2]_q}{q^2[n-2]_q}+\frac{q[2]_q[3]_q+[n]_q}{q^6[n-2]_q[n-3]_q}\right)x^2$$
$$+\frac{[n]_q+q(1+[2]_q)[n]_q}{q^5[n-2]_q[n-3]_q}x+\frac{[2]_q}{q^3[n-2]_q[n-3]_q},\quad n>3.$$

Lemma 3. The following equalities hold for $n\in N$ and $0\leq\alpha\leq\beta$ as

$$D_{n,\alpha,\beta}^{q}(1;x)=1;$$

$$D_{n,\alpha,\beta}^{q}(t;x)=\frac{[n]_q^2x+q[n]_q}{q^2([n]_q+\beta)[n-2]_q}+\frac{\alpha}{[n]_q+\beta};$$

$$D_{n,\alpha,\beta}^{q}(t^2;x)=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left[\frac{[n]_q(q[n]_q+1)}{q^6[n-2]_q[n-3]_q}x^2+\frac{[n]_q\{1+q(1+[2]_q)\}}{q^5[n-2]_q[n-3]_q}x+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}\right]+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}.\frac{[n]_qx+q}{q^2[n-2]_q}+\left(\frac{\alpha}{[n]_q+\beta}\right)^2.$$

Proof. The operators $D_{n,\alpha,\beta}^{q}(f(t);x)$ are well defined on the function $1,t,t^2$. Therefore for each $n\in N$ and $x\in[0,\infty)$, obviously $D_{n,\alpha,\beta}^{q}(1;x)=1$. Now

$$\begin{aligned}D_{n,\alpha,\beta}^{q}(t;x)&=[n-1]_q\sum_{v=0}^{\infty}p_{n,v}^q(x)\int_0^{\infty/A}p_{n,v}^q(t)\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)d_qt\\&=\left(\frac{[n]_q}{[n]_q+\beta}\right)D_n^q(t;x)+\left(\frac{\alpha}{[n]_q+\beta}\right)D_n^q(1;x)\\&=\frac{[n]_q^2x+q[n]_q}{q^2([n]_q+\beta)[n-2]_q}+\frac{\alpha}{[n]_q+\beta},\quad n>2\end{aligned}$$

and

$$\begin{aligned}&D_{n,\alpha,\beta}^{q}(t^2;x)\\&=[n-1]_q\sum_{v=0}^{\infty}p_{n,v}^q(x)\int_0^{\infty/A}p_{n,v}^q(t)\left(\frac{[n]_qt+\alpha}{[n]_q+\beta}\right)^2d_qt\\&=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2D_n^q(t^2;x)+\frac{2[n]_q\alpha}{([n]_q+\beta)^2}D_n^q(t;x)+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\&=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left[\left(1+\frac{[3]_q}{q^3[n-3]_q}+\frac{[2]_q}{q^2[n-2]_q}+\frac{q[2]_q[3]_q+[n]_q}{q^6[n-2]_q[n-3]_q}\right)x^2\right.\\&\left.+\frac{[n]_q+q(1+[2]_q)[n]_q}{q^5[n-2]_q[n-3]_q}x+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}\right]+\frac{2[n]_q\alpha}{([n]_q+\beta)^2}\frac{[n]_qx+q}{q^2[n-2]_q}\\&+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\end{aligned}$$

$$=\left(\frac{[n]_q}{[n]_q+\beta}\right)^2\left[\frac{[n]_q(q[n]_q+1)}{q^6[n-2]_q[n-3]_q}x^2+\frac{[n]_q\{1+q(1+[2]_q)\}}{q^5[n-2]_q[n-3]_q}x\right.$$

$$\left.+\frac{[2]_q}{q^3[n-2]_q[n-3]_q}\right]+\frac{2\alpha[n]_q}{([n]_q+\beta)^2}.\frac{[n]_q x+q}{q^2[n-2]_q}+\left(\frac{\alpha}{[n]_q+\beta}\right)^2,\quad n>3.$$

Remark 1. If we put $q=1$ and $\alpha=\beta=0,$ we get

$$D_n(t;x)=\frac{1+nx}{n-2},\quad n>2;$$

$$D_n(t-x;x)=\frac{1+2x}{n-2},\quad n>2;$$

$$D_n(t^2;x)=\frac{(n^2+n)x^2+4nx+2}{(n-2)(n-3)},\quad n>3;$$

$$D_n((t-x)^2;x)=\frac{2[(n+3)x^2+(n+3)x+1]}{(n-2)(n-3)},\quad n>3.$$

Lemma 4. The central moments of q-Baskakov-Durrmeyer-Stancu operators for $q\in(0,1)$ and $x\in[0,\infty)$ is

$$A^q_{n,\alpha,\beta,m}(x)=D^q_{n,\alpha,\beta}((t-x)^m;x)$$

$$=[n-1]_q\sum_{v=0}^{\infty}p^q_{n,v}(x)\int_0^{\infty}p^q_{n,v}(t)\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}-x\right)^m dt,$$

then we have

$$A^q_{n,\alpha,\beta,1}(x)=D^q_{n,\alpha,\beta}(t-x;x)=\frac{[n]_q([2]_q x+q)+q^2(\alpha-\beta x)[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}$$

$$A_{n,\alpha,\beta,2}^{q}(x) = D_{n,\alpha,\beta}^{q}((t-x)^2;x)$$
$$= \left[\frac{q[n]_q^4 + [n]_q^3}{q^6([n]_q+\beta)^2[n-2]_q[n-3]_q} - \frac{2[n]_q^2}{q^2([n]_q+\beta)[n-2]_q} + 1\right]x^2$$
$$+\left[\frac{[n]_q^3 + q(1+[2]_q)[n]_q^3}{q^5([n]_q+\beta)^2[n-2]_q[n-3]_q} + \frac{2\alpha[n]_q^2}{q^2([n]_q+\beta)^2[n-2]_q} - \frac{2[n]_q}{q([n]_q+\beta)[n-2]_q} - \frac{2\alpha}{[n]_q+\beta}\right]x + \frac{[2]_q[n]_q^2}{q^3([n]_q+\beta)^2[n-2]_q[n-3]_q}$$
$$+\frac{2\alpha[n]_q}{q([n]_q+\beta)^2[n-2]_q} + \left(\frac{\alpha}{[n]_q+\beta}\right)^2.$$

Proof. Using Lemma 3, we have

$$A_{n,\alpha,\beta,1}^{q}(x) = D_{n,\alpha,\beta}^{q}(t-x;x) = D_{n,\alpha,\beta}^{q}(t;x) - xD_{n,\alpha,\beta}^{q}(1;x)$$
$$= \frac{[n]_q^2 x + q[n]_q}{q^2([n]_q+\beta)[n-2]_q} + \frac{\alpha}{[n]_q+\beta} - x$$
$$= \frac{[n]_q([2]_q x + q) + q^2(\alpha-\beta x)[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}$$

and

$$A_{n,\alpha,\beta,2}^{q}(x) = D_{n,\alpha,\beta}^{q}((t-x)^2;x)$$
$$= D_{n,\alpha,\beta}^{q}(t^2;x) - 2xD_{n,\alpha,\beta}^{q}(t;x) + x^2 D_{n,\alpha,\beta}^{q}(1;x)$$
$$= \left[\frac{q[n]_q^4 + [n]_q^3}{q^6([n]_q+\beta)^2[n-2]_q[n-3]_q} - \frac{2[n]_q^2}{q^2([n]_q+\beta)[n-2]_q} + 1\right]x^2$$
$$+\left[\frac{[n]_q^3 + q(1+[2]_q)[n]_q^3}{q^5([n]_q+\beta)^2[n-2]_q[n-3]_q} + \frac{2\alpha[n]_q^2}{q^2([n]_q+\beta)^2[n-2]_q} - \frac{2[n]_q}{q([n]_q+\beta)[n-2]_q} - \frac{2\alpha}{[n]_q+\beta}\right]x + \frac{[2]_q[n]_q^2}{q^3([n]_q+\beta)^2[n-2]_q[n-3]_q}$$
$$+\frac{2\alpha[n]_q}{q([n]_q+\beta)^2[n-2]_q} + \left(\frac{\alpha}{[n]_q+\beta}\right)^2.$$

Lemma 5. For a given number $n > 3$ and $q \in (0,1)$, we have

$$D_{n,\alpha,\beta}^{q}((t-x)^2;x)\le\frac{7(1+\beta)[n]_q^2}{q^6([n]_q+\beta)[n-2]_q}\left(\varphi^2(x)+\frac{1}{([n]_q+\beta)[n-3]_q}\right),$$

where $\varphi^2(x)=x(1+x)$, for all $x[0,\infty)$.

Proof. From Lemma 4, we have

$$\begin{aligned}
&D_{n,\alpha,\beta}^{q}((t-x)^2;x)\\
&=\left[\frac{q[n]_q^4+[n]_q^3}{q^6([n]_q+\beta)^2[n-2]_q[n-3]_q}-\frac{2[n]_q^2}{q^2([n]_q+\beta)[n-2]_q}+1\right]x^2\\
&+\left[\frac{[n]_q^3+q(1+[2]_q)[n]_q^3}{q^5([n]_q+\beta)^2[n-2]_q[n-3]_q}+\frac{2\alpha[n]_q^2}{q^2([n]_q+\beta)^2[n-2]_q}-\right.\\
&\left.\frac{2[n]_q}{q([n]_q+\beta)[n-2]_q}-\frac{2\alpha}{[n]_q+\beta}\right]x+\frac{[2]_q[n]_q^2}{q^3([n]_q+\beta)^2[n-2]_q[n-3]_q}\\
&+\frac{2\alpha[n]_q}{q([n]_q+\beta)^2[n-2]_q}+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\
&=\left[\frac{q[n]_q^4+[n]_q^3}{q^6([n]_q+\beta)^2[n-2]_q[n-3]_q}-\frac{2[n]_q^2}{q^2([n]_q+\beta)[n-2]_q}+1\right](x^2+x)\\
&+\left[\frac{[n]_q^3+q(1+[2]_q)[n]_q^3}{q^5([n]_q+\beta)^2[n-2]_q[n-3]_q}+\frac{2\alpha[n]_q^2}{q^2([n]_q+\beta)^2[n-2]_q}-\right.\\
&\frac{2[n]_q}{q([n]_q+\beta)[n-2]_q}-\frac{2\alpha}{[n]_q+\beta}-\frac{q[n]_q^4+[n]_q^3}{q^6([n]_q+\beta)^2[n-2]_q[n-3]_q}\\
&\left.+\frac{2[n]_q^2}{q^2([n]_q+\beta)[n-2]_q}-1\right]x+\frac{[2]_q[n]_q^2}{q^3([n]_q+\beta)^2[n-2]_q[n-3]_q}\\
&+\frac{2\alpha[n]_q}{q([n]_q+\beta)^2[n-2]_q}+\left(\frac{\alpha}{[n]_q+\beta}\right)^2\\
&\le\frac{[n]_q^2}{q^6([n]_q+\beta)[n-2]_q[n-3]_q}[q[n]_q+1-2q^4[n-3]_q+q^6([n]_q+\beta)]\phi^2\\
&+\frac{[n]_q^2}{q^6([n]_q+\beta)^2[n-2]_q[n-3]_q}[q^3+q^4+2q^5\beta+q^6\beta^2]\\
&\le\frac{7(1+\beta)[n]_q^2}{q^6([n]_q+\beta)[n-2]_q}\left[\phi^2+\frac{1}{([n]_q+\beta)[n-3]_q}\right]
\end{aligned}$$

□

Definition 1. (Peetre's K-functional) Let us consider the space $C_B[0,\infty)$ of all the continuous and bounded functions f that is $f \in C_B[0,\infty)$ and endowed with the norm $\| f \| = \{| f(x) | : x \in [0,\infty)\}$, then the K-functional

$$K_2(f,\delta) = \inf_{g \in W^2} \{\| f - g \| + \delta \| g'' \|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. Also $\exists$ an absolute constant $C > 0$ such that $K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta})$, where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta}} \sup_{x \in [0,\infty)} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$.

Definition 2. (Rate of convergence) Let $B_{x^2}[0,\infty)$ be the set of all functions $f \in [0,\infty)$ satisfying the condition $| f(x) | \le M_f(1+x^2)$, M_f is a constant depending on f. We denote the subspace of all continuos functions by $C_{x^2}[0,\infty)$ belonging to $B_{x^2}[0,\infty)$. Again, we suppose $C^*_{x^2}[0,\infty)$ be the subspace of all the functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is defined as $\| f \|_{x^2} = \sup_{x\to\infty} \frac{|f(x)|}{1+x^2}$. We denote the usual modulus of continuity of f on the closed interval $[0,a]$ for $a > 0$, by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta} \sup_{x,t \in [0,a]} | f(t) - f(x) |.$$

We know that for a function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta) \to 0$.

3. Direct Estimates

In this section, we establish some direct and local approximation theorems connected with the operators $D^q_{n,\alpha,\beta}$ in simultaneous approximation.

Local Approximation Theorem

Theorem 1. For $q\in(0,1)$ and $n\geq 4,$ we have

$$|D_{n,\alpha,\beta}^{q}(f;x)-f(x)|\leq C\omega_2(f,\delta_n(x))+\omega\left(f,\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}\right)$$

where for all $x\in[0,\infty)$ and $f\in C_B[0,\infty),$ C is a positive constant; and

$$\delta_n^2(x)=A_{n,\alpha,\beta,2}^{q}(x)+(A_{n,\alpha,\beta,1}^{q}(x))^2.$$

Proof. We define the auxiliary operators $\overline{D}_{n,\alpha,\beta}^{q}$ for $x\in[0,\infty)$ as

$$\overline{D}_{n,\alpha,\beta}^{q}(f;x)=D_{n,\alpha,\beta}^{q}(f;x)+f(x)-f\left(x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}\right). \quad (5)$$

From Lemma 3, the operators $\overline{D}_{n,\alpha,\beta}^{q}$ is observed to be linear and preserving the linear functions as

$$\overline{D}_{n,\alpha,\beta}^{q}(t-x;x)=0. \quad (6)$$

By Taylor's expansion of a function $g\in W^2$ as

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-w)g''(w)dw,x,t\in[0,\infty)$$

and (6), we obtain

$$\overline{D}_{n,\alpha,\beta}^{q}(g;x)=g(x)+\overline{D}_{n,\alpha,\beta}^{q}\left(\int_x^t(t-w)g''(w)dw;x\right)$$

Hence from (5), one get

$$|\overline{D}_{n,\alpha,\beta}^{q}(g;x)-g(x)|$$

$$\leq\left|D_{n,\alpha,\beta}^{q}\left(\int_x^t(t-w)g''(w)dw;x\right)\right|+\left|\int_x^{x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}}\right.$$

$$\left(x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}-w\right) g''(w)dw\Bigg|$$

$$\leq D_{n,\alpha,\beta}^q\left(\int_x^t |t-w|\,|g''(w)|\,dw,x\right)+\int_x^{x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}}$$

$$\left|x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}-w\right| |g''(w)|\,dw$$

$$\leq\left[D_{n,\alpha,\beta}^q((t-x)^2,x)+\left(D_{n,\alpha,\beta}^q(t-x;x)\right)^2\right]\|g''\|=\delta_n^2(x)\|g''\|. \tag{7}$$

Now from (5), we have

$$|\overline{D}_{n,\alpha,\beta}^q(f;x)|\leq|D_{n,\alpha,\beta}^q(f;x)|+2\|f\|\leq\|f\|\,D_{n,\alpha,\beta}^q(1;x)+2\|f\|\leq 3\|f\|. \tag{8}$$

Therefore from (5), (7) and (8),

$$\begin{aligned}
&|D_{n,\alpha,\beta}^q(f;x)-f(x)|\\
&=|\overline{D}_{n,\alpha,\beta}^q(f-g,x)-(f-g)(x)|+|\overline{D}_{n,\alpha,\beta}^q(g,x)-g(x)|\\
&+\left|f\left(x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}\right)-f(x)\right|\\
&\leq 4\|f-g\|+\delta_n^2(x)\|g''\|\\
&+\left|f(x+\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q})-f(x)\right|.
\end{aligned}$$

Taking infimum overall $g\in W^2$ on RHS and then using Peetre's K-functional defined in the previous section, we have

$$\begin{aligned}
|D_{n,\alpha,\beta}^q(f;x)-f(x)|&\leq CK_2(f,\delta_n^2(x))+\omega\left(f,\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}\right)\\
&=C\omega_2(f,\delta_n(x))+\omega\left(f,\frac{\{[n]_q^2-q^2([n]_q+\beta)[n-2]_q\}x+q[n]_q+\alpha q^2[n-2]_q}{q^2([n]_q+\beta)[n-2]_q}\right).
\end{aligned}$$

Hence the proof of theorem is completed.

Theorem 2. Let $f \in C_{x^2}[0,\infty), q = q_n \in (0,1)$ such that $q_n \to 1$ as $n \to \infty$ and $\omega_{a+1}(f,\delta)$ be its modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a > 0$. Then for every $n > 3$, we have

$$\| D^q_{n,\alpha,\beta}(f;x) - f(x) \|_{C[0,a]} \le \frac{R[n]_q}{q^6([n]_q + \beta)[n-3]_q} + 2\omega_{a+1}\left(f, \sqrt{\frac{[n]_q R}{q^6([n]_q + \beta)[n-3]_q}} \right),$$

where $R = 42(1+\beta)M_f(1+a^2)(1+a+a^2)$.

Proof. For $x \in [0,a]$ and $t > a+1$, as $t - x > 1$, we have

$$| f(t) - f(x) | \le M_f(2 + x^2 + t^2) \le M_f(2 + 3x^2 + (t-x)^2) \le 6M_f(1+a^2)(t-x)^2. \tag{9}$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$| f(t) - f(x) | \le \omega_{a+1}(f, | t-x |) \le \left(1 + \frac{| t-x |}{\delta} \right) \omega_{a+1}(f,\delta), \quad \delta > 0 \tag{10}$$

From (9) and (10), we have

$$| f(t) - f(x) | \le 6M_f(1+a^2)(t-x)^2 + \left(1 + \frac{| t-x |}{\delta} \right) \omega_{a+1}(f,\delta) \tag{11}$$

for $x \in [0,a]$ and $t \ge 0$. Hence

$$\begin{aligned} | D^q_{n,\alpha,\beta}(f;x) - f(x) | &\le D^q_{n,\alpha,\beta}(| f(t) - f(x) |; x) \\ &\le 6M_f(1+a^2) D^q_{n,\alpha,\beta}((t-x)^2; x) \\ &+ \omega_{a+1}(f,\delta) \left(1 + \frac{1}{\delta} D^q_{n,\alpha,\beta}((t-x)^2; x)^{1/2} \right). \end{aligned}$$

Using Schwarz inequality and Lemma 4,

$$
\begin{aligned}
&| D_{n,\alpha,\beta}^{q}(f;x)-f(x)| \\
&\leq \frac{42(1+\beta)M_f(1+a^2)[n]_q^2}{q^6([n]_q+\beta)[n-2]_q}\left(\phi^2(x)+\frac{1}{([n]_q+\beta)[n-3]_q}\right) \\
&+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{7[n]_q^2(1+\beta)}{q^6([n]_q+\beta)[n-2]_q}\left(\phi^2(x)+\frac{1}{([n]_q+\beta)[n-3]_q}\right)}\right) \\
&\leq \frac{R[n]_q}{q^6([n]_q+\beta)[n-3]_q}+\omega_{a+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\frac{R[n]_q}{q^6([n]_q+\beta)[n-3]_q}}\right).
\end{aligned}
$$

Taking $\delta=\sqrt{\frac{R[n]_q}{q^6([n]_q+\beta)[n-3]_q}}$, we obtain the assertion of our theorem.

Corollary 1. If $f\in Lip_M\theta$ on $[0,a+1]$, then for $n>3$

$$
\| D_{n,\alpha,\beta}^{q}(f;x)-f(x)\|_{C[0,a]}\leq (1+2M)\sqrt{\frac{R[n]_q}{q^6([n]_q+\beta)[n-3]_q}}.
$$

Proof. For n to be sufficiently large, we have

$$
\frac{R[n]_q}{q^6([n]_q+\beta)[n-3]_q}\leq\sqrt{\frac{R[n]_q}{q^6([n]_q+\beta)[n-3]_q}}
$$

Since $\lim_{n\to\infty}[n-3]_q=\infty$, by $f\in Lip_M\theta$ we obtain the required corollary.

Weighted approximation theorem After that we discuss about the weighted approximation theorem, which holds true on $[0,\infty)$.

Theorem 3. Let $q\equiv q_n$ satisfies $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Then for each $f\in C_{x^2}^{*}[0,\infty)$, we have

$$
\lim_{n\to\infty}\| D_{n,\alpha,\beta}^{q}(f;x)-f(x)\|_{x^2}=0
$$

Proof: Using the theorem in [3], the following three conditions are sufficient to varify

$$
\lim_{n\to\infty}\| D_{n,\alpha,\beta}^{q}(t^k,x)-x^k\|_{x^2}=0, k=0,1,2. \tag{12}
$$

as $D_n^{q_n}(1,x)=1,$ the first condition $k=0$ of (12) is fulfilled. Therefore by Lemma 3, for $n>2$ we have

$$\begin{aligned}\| D_{n,\alpha,\beta}^{q_n}(t;x)-x\|_{x^2} &= \sup_{x\in[0,\infty)} \frac{|D_{n,\alpha,\beta}^{q_n}(t;x)-x|}{1+x^2} \\ &\le \frac{[n]_{q_n}[2]_{q_n}-q_n{}^2\beta[n-2]_{q_n}}{q_n{}^2([n]_{q_n}+\beta)[n-2]_{q_n}} \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &+ \frac{[n]_{q_n}+q_n\alpha[n-2]_{q_n}}{q_n([n]_{q_n}+\beta)[n-2]_{q_n}} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\le \frac{[n]_{q_n}[2]_{q_n}-q_n{}^2\beta[n-2]_{q_n}}{q_n{}^2([n]_{q_n}+\beta)[n-2]_{q_n}} + \frac{[n]_{q_n}+q_n\alpha[n-2]_{q_n}}{q_n([n]_{q_n}+\beta)[n-2]_{q_n}},\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \| D_{n,\alpha,\beta}^{q_n}(t;x)-x\|_{x^2}=0$$

Thus the second condition of (12) is also varified i.e. for $k=1$ as $n\to\infty.$ Similarly for $n>3,$ we can take

$$\begin{aligned}&\| D_{n,\alpha,\beta}^{q_n}(t^2;x)-x^2\|_{x^2} \\ &= \left(\frac{q_n[n]_{q_n}^4+[n]_{q_n}^3}{q_n{}^6([n]_{q_n}+\beta)^2[n-2]_{q_n}[n-3]_{q_n}}-1\right) \sup_{x\in[0,\infty)} \frac{x^2}{1+x^2} \\ &+ \left(\frac{[n]_{q_n}^3\{1+q_n(1+[2]_{q_n})+2\alpha q_n{}^3[n]_{q_n}^2[n-3]_{q_n}\}}{q_n{}^5([n]_{q_n}+\beta)^2[n-2]_{q_n}[n-3]_{q_n}}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &+ \left(\frac{[2]_{q_n}[n]_{q_n}^2+2q_n{}^2\alpha[n]_{q_n}[n-3]_{q_n}}{q_n{}^3([n]_{q_n}+\beta)^2[n-2]_{q_n}[n-3]_{q_n}} + \left(\frac{\alpha}{[n]_{q_n}+\beta}\right)^2\right) \sup_{x\in[0,\infty)} \frac{1}{1+x^2},\end{aligned}$$

which also implies that

$$\lim_{n\to\infty} \| D_{n,\alpha,\beta}^{q_n}(t^2;x)-x\|_{x^2}=0.$$

Hence the proof of theorem is completed.

Theorem 4. If $q \equiv q_n$ satisfies $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$, for each $f \in C_{x^2}[0,\infty)$ and $\theta > 0$, we have

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{|D_{n,\alpha,\beta}^{q_n}(f;x) - f(x)|}{(1+x^2)^{1+\theta}} = 0.$$

Proof: For fixed $x_0 > 0$,

$$\begin{aligned}
&\sup_{x\in[0,\infty)} \frac{|D_{n,\alpha,\beta}^{q_n}(f;x) - f(x)|}{(1+x^2)^{1+\theta}} \\
&\leq \sup_{x\leq x_0} \frac{|D_{n,\alpha,\beta}^{q_n}(f;x) - f(x)|}{(1+x^2)^{1+\theta}} + \sup_{x>x_0} \frac{|D_{n,\alpha,\beta}^{q_n}(f;x) - f(x)|}{(1+x^2)^{1+\theta}} \\
&= \| D_{n,\alpha,\beta}^{q_n}(f) - f \|_{C[0,x_0]} + \| f \|_{x_2} \sup_{x>x_0} \frac{|D_{n,\alpha,\beta}^{q_n}(1+t^2;x)|}{(1+x^2)^{1+\theta}} + \sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\theta}}.
\end{aligned}$$

First term of the above inequality tends to zero by Theorem 2. From Lemma 2 for a fixed $x_0 > 0$ it can easily be seen that $n \to \infty$ implies

$$\sup_{x>x_0} \frac{|D_{n,\alpha,\beta}^{q_n}(1+t^2;x)|}{(1+x^2)^{1+\theta}} \to 0.$$

Therefore we can choose $x_0 > 0$ too large to be $\sup_{x>x_0} \frac{|f(x)|}{(1+x^2)^{1+\theta}}$ small enough. Thus the proof of theorem is done.

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