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Two Problems of Rhythmical Manufacturing Process

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Abstract

In this paper, a relationship between two problems of manufacturing process planning with an unstable (fluctuating) sequence of raw materials supply is considered. The first problem is a problem of smoothing of the initial sequence using a stock of limited volume for raw materials. This problem is stated and solved as a problem of convex programming subject to constraints produced by the presence of the stock. The second problem (which is not classical) is to find such a plan that satisfies the constraints and has the least number of changes of manufacturing process intensity. It is shown that the optimal plan to the first problem may be a solution to the second problem under certain conditions, in general case it gives the possibility to determine a lower bound of changes for each feasible plan.

Keywords: smoothing, convex programming, active constraints, number of changes.

Introduction

The notion of smoothing is very wide, and the problem of smoothing has both many types and many different applications (see [1-4]). The choice of the type of smoothing is determined by concrete purposes of investigation. Usually, the process of smoothing consists of replacing each data point of an initial sequence by some kind of average of surrounding data points. In the paper [5] is considered a problem of optimal smoothing with constraints, reduced to quadratic programming. We propose also a type of optimal smoothing for a problem of manufacturing process planning with a fluctuating supply of raw material. A presence of a stock of limited volume generates a set D of feasible smoothed vectors. The problem of optimal smoothing is reduced to a problem of convex separable programming [6]. Simple necessary and sufficient conditions of optimality are received. A decomposition algorithm to calculate optimal vector

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$X^0 \in D$ is suggested.

Another type of optimal smoothed vectors is such a vector $X^* \in (x_1^*, x_2^*, \ldots, x_n^*) \in D$ that has the least number of the indexes $i \in \{1,2,\ldots,n-1\}$, where $x_i^* \neq x_{i+1}^*$. It is shown [7, 8] that in a particular case the optimal vector X^0 coincides with one of vectors X^* (as a rule, there is a set of such vectors). In general case a lower bound of the least number of changes where $x_i^* \neq x_{i+1}^*$ is received. A heuristic algorithm is elaborated to calculate optimal vectors with the least number of changes and some numerical examples are presented to illustrate these results.

Model and Analysis

Let an initial sequence $\{P_i\}, i = \overline{1,n},$ be the input of raw materials to the stock of the volume α and a smoothed sequence $\{x_i\}, i = \overline{1,n},$ be the output of raw materials to be manufactured. The components of each smoothed feasible vector $X = (x_1, x_2, \ldots, x_n)$ have to satisfy the restrictions

$$\sum_{i=1}^{j} P_i - \alpha \leq \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} P_i, j = \overline{1,n-1}, \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} P_i,$$

where the parameter α is a factor of smoothing. Let us designate $A_j = \sum_{i=1}^{j} P_i - \alpha$, $B_j = \sum_{i=1}^{j} P_i$, $j = \overline{1,n-1}$, $A_n = B_n = \sum_{i=1}^{n} P_i$. The problem of optimal smoothing can be stated [7] as a problem of convex programming: find a vector $X^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in D$, that minimizes the function

$$F(X) = \sum_{i=1}^{n} f(x_i), \tag{1}$$

where $f(x)$ is a continuous strictly convex function, and the feasible set D.

$$D = \left\{X \in R^n : A_j \leq \sum_{i=1}^{j} x_i \leq B_j, A_j < B_j, j = \overline{1,n-1}, \sum_{i=1}^{n} x_i = A_n = B_n\right\} \tag{2}$$

is a special set of R^n with lower (A_j) and upper (B_j) constraints.

It's evident that an optimal vector exists because the function (1) is continuous and the set D is limited and closed.

Theorem 1. A vector $X^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in D$ minimizes the function $F(X)$ if and only if every pair of its components x_k^0 and x_j^0 ($k > j$) satisfies one of the following conditions:

A) $x_k^0 = x_j^0$;

B) $x_k^0 > x_j^0$, and there exists such a number $m \in \{j, j+1, \ldots k-1\}$ that $\sum_{i=1}^{i=m} x_i^0 = B_m$;

C) $x_k^0 < x_j^0$, and there exists such a number $l \in \{j, j+1, \dots k-1\}$ that $\sum_{i=1}^{i=l} x_i^0 = A_l$.

Necessity. Notice that for any strictly convex function $f(x)$ its difference $\Delta(x) = f(x+h) - f(x)$ is strictly increasing with $h > 0$. We assume the conditions A), B), C) to be false at least for one pair of components x_k^0 and x_j^0 $(k > j)$. That is $x_k^0 \neq x_j^0$ and

a) if $x_k^0 > x_j^0$, then $\sum_{i=1}^{i=m} x_i^0 < B_m$ for every $m \in \{j, j+1, \dots k-1\}$;

b) if $x_k^0 < x_j^0$, then $\sum_{i=1}^{i=l} x_i^0 > A_l$ for every $l \in \{j, j+1, \dots k-1\}$.

For the case a) we'll construct the new vector $X(h) = \left(x_1^0, \dots, x_j^0 + h, x_{j+1}^0, \dots, x_k^0 - h, x_{k+1}^0, \dots x_n^0\right)$. If h is a positive rather little number, then this vector will be feasible. Moreover the difference $F\big(X(h)\big) - F(X^0) = f\big(x_j^0 + h\big) - f\big(x_j^0\big) + f(x_k^0 - h) - f\big(x_k^0\big) = \Delta\big(x_j^0\big) - \Delta\big(x_k^0 - h\big)$ will be negative if $h < x_k^0 - x_j^0$. By the same way we'll receive the contradiction to optimality of the vector X^0 for the case b).

Sufficiency. Assume every pair of a vector X^0 to satisfy one of the conditions A),B),C), but there exists another feasible vector $Y^0 = (y_1^0, y_2^0, \dots, y_n^0)$ such that minimizes the function (1) in the set D. It means that every pair of the components y_k^0 and y_j^0 $(k > j)$ satisfies one of the conditions A),B),C). We'll prove that these vectors have to be the same. Indeed, assume $y_1^0 = x_1^0, y_2^0 = x_2^0, \dots, y_{j-1}^0 = x_{j-1}^0$, but $y_j^0 > x_j^0$ (the case $y_j^0 < x_j^0$ may be considered by the same way). Since $\sum_{i=1}^{i=n} y_i^0 = \sum_{i=1}^{i=n} x_i^0 = B_n$ there exist such a number $k > j$ that the following relationships are held:

$$y_j^0 > x_j^0, y_{j+1}^0 \geq x_{j+1}^0, y_{j+1}^0 \geq x_{j+2}^0, \dots, y_{k-1}^0 \geq x_{k-1}^0, y_k^0 < x_k^0. \tag{3}$$

In the case $y_j^0 \leq y_k^0$ we'll receive the following inequalities $x_k^0 > y_k^0 \geq y_j^0 > x_j^0$. In accordance with the condition B) there exists such a number $m \in \{j, j+1, \dots k-1\}$ that $\sum_{i=1}^{i=m} x_i^0 = B_m$. It means that the components of the vector Y^0 are to satisfy the inequality $\sum_{i=1}^{i=m} y_i^0 > B_m$, that is this vector is not feasible!

In the other case when $y_j^0 > y_k^0$ in accordance with the condition C) there exists such a number $l \in \{j, j+1, \dots k-1\}$, that $\sum_{i=1}^{i=l} x_i^0 = A_l$. It means that the components of the vector X^0 are to satisfy the inequality $\sum_{i=1}^{i=l} x_i^0 < A_l$, that contradicts to the fact $X^0 \in D$.

So the vectors X^0 and Y^0 coincide.

Corollary 1. *A vector* $X^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ *minimizes the function* $F(X)$ *if and only if every pair of its components* x_k^0 *and* x_{k+1}^0 *satisfies one of the following conditions:*

A) $x_{k+1}^0 = x_k^0$*;*

B) $x_{k+1}^0 > x_k^0$ *and* $\sum_{i=1}^{i=k} x_i^0 = B_k$;

C) $x_{k+1}^0 < x_k^0$ *and* $\sum_{i=1}^{i=k} x_i^0 = A_k$.

Corollary 2. *A vector* $X^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ *minimizes the function* $F(X)$ *if and only if for each number* $k \in \{1,2,\dots,n-1\}$ *one of the following conditions is held:*

1) *if* $\sum_{i=1}^{k} x_i^0 \in (A_k, B_k)$ *then* $x_k^0 = x_{k+1}^0$;
2) *if* $\sum_{i=1}^{k} x_i^0 = A_k$ *then* $x_k^0 \geq x_{k+1}^0$;
3) *if* $\sum_{i=1}^{k} x_i^0 = B_k$ *then* $x_k^0 \leq x_{k+1}^0$.

Theorem 2. Let $R_+^n = \{X \in R^n : x_i \geq 0, i = \overleftarrow{1,n}\}$. These three assertions below are equivalent:

1) the set $D \cap R_+^n \neq \emptyset$;

2) the inequalities $B_1 \geq 0, B_j \geq max\{0, A_1, \dots, A_{j-1}\}, j = \overleftarrow{2,n},$ are held;

3) all the components of the optimal vector $X^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ are nonnegative.

Let us prove that the assertion 1) implies the assertion 2). Indeed, assume there exist such a pair of indexes m and k $(k > m)$ that $B_k < A_m$ (if $m = 0$ then $A_0 = 0$). In this case we obtain the following inequality

$$\sum_{i=m+1}^{i=k} x_i = \sum_{i=1}^{i=k} x_i - \sum_{i=1}^{i=m} x_i < B_k - A_m < 0. \tag{4}$$

It means that at least one component of any feasible vector $X \in D$ is to be negative.

Let us prove that the assertion 2) implies the assertion 3). Assume there exist negative components of the optimal vector $X^0 \in D$. Its evident there are positive components as well because $\sum_{i=1}^{i=n} x_i^0 = B_n \geq 0$. Let x_j^0 be negative component with the least index. There are two possible alternatives:

a) all positive components follow x_j^0;

b) there is at least one positive component $x_k^0, k < j$.

In the first case the following inequalities

$$x_1^0 = 0,,,, x_{j-1}^0 = 0, x_j^0 < 0, x_{j+1}^0 \leq 0, \dots, x_{m-1}^0 \leq 0, x_m^0 > 0, \dots \tag{5}$$

are held. In accordance with the corollary 1 $\sum_{i=1}^{i=m-1} x_i^0 = B_{m-1} < 0$ because $x_m^0 > x_{m-1}^0$.

In the second case the following inequalities

$$x_1^0 \geq 0, \dots, x_{k-1}^0 \geq 0, x_k^0 > 0, x_{k+1}^0 = 0, \dots, x_{j-1}^0 = 0, x_j^0 < 0, \dots \tag{6}$$

are held. In accordance with the corollary 1 $\sum_{i=1}^{i=k} x_i^0 = A_k$ because $x_{k+1}^0 < x_k^0$. If all components

$x_i^0 \le 0, i = \overleftarrow{j+1, n}$, then $\sum_{i=k+1}^{i=n} x_i^0 = B_n - A_k < 0$. If there exists a component x_{l+1}^0 such that $x_j^0 < 0, x_{j+1}^0 \le 0, \dots, x_l^0 \le 0, x_{l+1}^0 > 0, \dots$ then $\sum_{i=1}^{i=l} x_i^0 = B_l$, because $x_{l+1}^0 > x_l^0$. It means that $\sum_{i=k+1}^{i=l} x_i^0 = B_l - A_k < 0$.

Obviously, the assertion 3 implies the assertion 1.

Definition 1. The restriction $A_j \le \sum_{i=1}^{j} x_i$ ($\sum_{i=1}^{j} x_i \le B_j$) is said to be an active constraint for the optimal vector $X^0 = (x_1^0, x_2^0, \dots, x_n^0)$ if $A_j = \sum_{i=1}^{j} x_i^0$ and $x_j^0 > x_{j+1}^0$ ($\sum_{i=1}^{j} x_i^0 = B_j$ and $x_j^0 < x_{j+1}^0$). This definition a little differs from the usual definition of active constraints [9].

The following theorem gives the possibility to determine step by step all active constraints and readily find the optimal vector X^0 by the decomposition of the initial problem into similar problems of less dimension.

Theorem 3. *Let* $\mu_i = B_i - i\frac{B_n}{n}$ *and* $\nu_i = i\frac{B_n}{n} - A_i, i = \overleftarrow{1, n-1}$, then

a) *if* $\mu_i \ge 0, \nu_i \ge 0, i = \overleftarrow{1, n-1}$, *then all the components of optimal vector are equal to* $\frac{B_n}{n}$;

b) *if* $\mu_k = min\mu_i < 0$, *then the components of optimal vector satisfy the condition* $\sum_{i=1}^{i=k} x_i^0 = B_k$;

c) *if* $\nu_l = min\nu_i < 0$, *then the components of optimal vector satisfy the condition* $\sum_{i=1}^{i=l} x_i^0 = A_l$.

The assertion *a)* results from the theorem 1.

Assume $\mu_k = min\mu_i = B_k - k\frac{B_n}{n} < 0$, but $\sum_{i=1}^{i=k} x_i^0 = C_k < B_k$. In accordance with the corollary 2 we have the inequality $x_k^0 \ge x_{k+1}^0$. Let us consider two cases depending on the value of the component x_k^0.

1) In the case $x_k^0 \ge \frac{B_n}{n}$ the inequalities $\sum_{i=1}^{i=k} x_i^0 = C_k < B_k < k\frac{B_n}{n}$ mean that there exists such a number $j < k$ that $x_j^0 < \frac{B_n}{n}, x_{j+1}^0 \ge \frac{B_n}{n}, \dots, x_k^0 \ge \frac{B_n}{n}$. In accordance with the corollary 1 it means that $\sum_{i=1}^{i=j} x_i^0 = B_j$. Estimating the difference $\mu_k - \mu_j = B_k - k\frac{B_n}{n} - B_j + j\frac{B_n}{n} > C_k - B_j - (k-j)\frac{B_n}{n} = \sum_{i=j+1}^{i=k} x_i^0 - (k-j)\frac{B_n}{n} \ge 0$

   We receive the contradiction with the definition of the value $\mu_k = min\mu_i$.

2) In the case $x_{k+1}^0 \le x_k^0 < \frac{B_n}{n}$ the inequalities $\sum_{i=k+1}^{i=n} x_i^0 = B_n - C_k > B_n - B_k > B_n - k\frac{B_n}{n} = (n-k)\frac{B_n}{n}$ mean that there exists such a number $l > k$ that $x_{k+1}^0 < \frac{B_n}{n}, x_{k+2}^0 < \frac{B_n}{n}, \dots, x_l^0 <$

$\frac{B_n}{n}, x_{l+1}^0 \geq \frac{B_{n.}}{n}$. In accordance with the corollary 1 it means that $\sum_{i=1}^{i=l} x_i^0 = B_l$. Estimating the difference $\mu_l - \mu_k = B_l - l\frac{B_n}{n} - B_k + k\frac{B_n}{n} < B_l - C_k - (l-k)\frac{B_n}{n} = \sum_{i=k+1}^{i=l} x_i^0 - (l-k)\frac{B_n}{n} < 0$

we receive the contradiction with the definition of the value $\mu_k = min\mu_i$.

Corollary 3. The structure of the optimal vector $X^0 = (x_1^0, x_2^0, \dots, x_n^0) \in D$ is completely determined by the presence of p $(0 \leq p \leq n-1)$ active constraints (notice that in the simple case, when $p = 0$ the optimal vector is absolutely smoothed: $x_i^0 = {B_n}/{n}, i = \overline{1,n}$). Namely, let $s_1, s_2, \dots, s_p$ $(0 < s_1 < s_2 < \cdots s_p < n)$ be the numbers of the active constraints $\sum_{i=1}^{s_l} x_i^0 = M_{s_l}$, $l = \overline{1,p}$. Then $x_i^0 = z_l = \frac{M_{s_l} - M_{s_{l-1}}}{s_l - s_{l-1}}$ for $i = \overline{s_{l-1}+1, s_l}$, $l = \overline{1, p+1}$ where we introduce the notations $s_0 = 0, M_0 = 0, s_{p+1} = n, M_n = A_n = B_n$.

In the case when lower active constraints and upper active constraints are strictly following one another, the optimal vector X^0 at the same time has the minimal number p of changes [7, 8]. Generally, the set of p active constraints consists of some number q $(q \leq p)$ groups of the same (either lower or upper) type consecutive constraints. Let $M_{j_k}, k = \overline{1,q}, j_k < j_{k+1}$, be representatives of these groups, and $y_k = \frac{M_{j_k} - M_{j_{k-1}}}{j_k - j_{k-1}}, k = \overline{1, q+1}, (j_0 = 0,\ M_{j_0} = 0, j_{q+1} = n,\ M_n = \mathrm{A_n} = B_n)$.

Theorem 4. Let every pair of the values y_k and y_{k+1}, $k = \overline{1,q}$, satisfies the inequality $y_k < y_{k+1}$ $(y_k > y_{k+1})$ if the active constraint M_{j_k} is upper (lower) constraint. Then q will be the least possible number of changes for any feasible vector $X \in D$.

Let us introduce the subsets $J_{k-1,k+m} = \{j_{k-1}+1, j_{k-1}+2, \dots, j_{k+m}-1, j_{k+m}\}$, $k \in \{1, .. q+1\}, m \in \{0,1,\dots, q-k+1\}$, of the set $J_{0,q+1} = \{1,2, \dots n\}$. The number $(m+1)$ we'll name the length of the subset $J_{k-1,k+m}$.

Lemma. Every vector $X \in D$ under the assumptions of the theorem 3 will have at least *m* changes at any subset $J_{k-1,k+m}$ of the length $(m+1)$.

The proof of this lemma is realized by means of mathematical induction.

At first we'll prove that every feasible vector $X \in D$ has at least one change at any subset $J_{k-1,k+1}$ of the length 2. Suppose there exist such a vector $X = (x_1, \dots, x_n) \in D$ and such a subset $J_{k-1,k+1}$ of the length 2, that $x_i = z_k$ = const for all $i \in J_{k-1,k+1}$. It's not difficult to receive the following inequalities for the components of any feasible vector $X \in D$:

$$A_{j_{k+1}} - B_{j_k} \leq \sum_{i=j_k+1}^{i=j_{k+1}} x_i \leq B_{j_{k+1}} - A_{j_k}, k \in \{0,1,\dots,q\}. \quad (7)$$

It means that the constant z_k has to satisfy the inequalities

$$\frac{A_{j_k} - B_{j_{k-1}}}{j_k - j_{k-1}} \leq z_k \leq \frac{B_{j_k} - A_{j_{k-1}}}{j_k - j_{k-1}}, \quad (8)$$

$$\frac{A_{j_{k+1}} - B_{j_k}}{j_{k+1} - j_k} \leq z_k \leq \frac{B_{j_{k+1}} - A_{j_k}}{j_{k+1} - j_k}. \quad (9)$$

These inequalities can rewritten as follows

$$\frac{A_{j_k} - B_{j_{k-1}}}{j_k - j_{k-1}} \leq z_k \leq \frac{B_{j_{k+1}} - A_{j_k}}{j_{k+1} - j_k}, \quad (10)$$

$$\frac{A_{j_{k+1}} - B_{j_k}}{j_{k+1} - j_k} \leq z_k \leq \frac{B_{j_k} - A_{j_{k-1}}}{j_k - j_{k-1}}. \quad (11)$$

If the active constraint M_{j_k} is an upper one ($M_{j_k} = B_{j_k}$) then the inequality

$$y_k = \frac{B_{j_k} - A_{j_{k-1}}}{j_k - j_{k-1}} < y_{k+1} = \frac{A_{j_{k+1}} - B_{j_k}}{j_{k+1} - j_k} \quad (12)$$

has to be valid in accordance with the assumptions of the theorem 3. But it contradicts the inequality (10). If the active constraint M_{j_k} is a lower one ($M_{j_k} = A_{j_k}$) then the inequality

$$y_k = \frac{A_{j_k} - B_{j_{k-1}}}{j_k - j_{k-1}} > y_{k+1} = \frac{B_{j_{k+1}} - A_{j_k}}{j_{k+1} - j_k} \quad (13)$$

has to be valid in accordance with the assumptions of the theorem 3. But it contradicts the inequality (10). So we have proved that every feasible vector $X \in D$ has at least one change at any subset $J_{k-1,k+1}$ of the length 2.

Now we'll prove the following assertion: if any feasible vector $X \in D$ has at least (t - 1) changes at any subset $J_{k-1,k+t-1}$ of the length t, $t = 2,3,\dots,m$, then any feasible vector $X \in D$ will have at least m changes at any subset $J_{k-1,k+m}$ of the length $m + 1$.

Suppose there exists a feasible vector $X = (x_1, \dots, x_n) \in D$ having only $(m - 1)$ changes at a subset $J_{k-1,k+m}$ of the length $(m + 1)$. Firstly, it means there are not changes at the subsets $J_{k-1,k}$ and $J_{k+m-1,k+m}$. Secondly, there is only one change at each subset $J_{k+r-1,k+r}$, $r = \overline{1, m-1}$, of the length 1. Really, otherwise we'll receive a contradiction with our assumption about the subsets of the length t, t = 2, 3,…, m.

Let $l_r \in J_{k+r-1,k+r}$, $r = \overline{1, m-1}$, be the numbers of changes, i.e. where $x_{l_r} \neq x_{l_r+1}$. So we'll have the following values of the components of the vector $X = (x_1, \dots, x_n) \in D$: $x_i = z_k = const$ for $i = \overline{j_{k-1}+1, l_1}$; $x_i = z_{k+r} = const$ for $i = \overline{l_r, l_{r+1}}$, $r = \overline{1, m-2}$; $x_i = z_{k+m-1} = const$ for $i = \overline{l_{m-1}, j_{k+m}}$.

We'll consider in details the case when the active constraint M_{j_k} is an upper one, i.e. $M_{j_k} = B_{j_k}$. The other case when the active constraint M_{j_k} is a lower one can be considered by analogy.

So the inequality (12) is valid, and for the components x_i at the subset $J_{k-1,k+1}$ we'll have

$x_i = z_k$ for $i = \overline{j_{k-1}+1, l_1}$, and $x_i = z_{k+1}$ for $i = \overline{l_1+1, j_{k+1}}$. Let us prove that $z_k < z_{k+1}$.

Indeed, the inequalities (7) imply the following estimates for the values z_k and z_{k+1}:

$$\frac{A_{j_k} - B_{j_{k-1}}}{j_k - j_{k-1}} \le z_k \le \frac{B_{j_k} - A_{j_{k-1}}}{j_k - j_{k-1}}, \quad A_{j_{k+1}} - B_{j_k} \le (l_1 - j_k) z_k + (j_{k+1} - l_1) z_{k+1} \le B_{j_{k+1}} - A_{j_k}.$$

If we suppose that $z_k \ge z_{k+1}$, then we'll receive the following estimates for the value z_k:

$$\frac{A_{j_{k+1}} - B_{j_k}}{j_{k+1} - j_k} \le z_k \le \frac{B_{j_k} - A_{j_{k-1}}}{j_k - j_{k-1}}.$$

This inequality contradicts the inequality (12). So the inequality $z_k < z_{k+1}$ is valid and for the value z_{k+1} we'll receive the lower estimate:

$$z_{k+1} \ge \frac{A_{j_{k+1}} - B_{j_k}}{j_{k+1} - j_k}. \tag{14}$$

For the next subset $J_{k+1,k+2}$ the inequalities (7) involve the following estimates of the values z_{k+1} and z_{k+2}:

$$A_{j_{k+2}} - B_{j_{k+1}} \le (l_2 - j_{k+1}) z_{k+1} + (j_{k+2} - l_2) z_{k+2} \le B_{j_{k+2}} - A_{j_{k+1}}.$$

Taking into account that $M_{j_{k+1}} = A_{j_{k+1}}$ we'll prove that $z_{k+1} > z_{k+2}$. Indeed, otherwise the value z_{k+1} will have the upper estimate

$$z_{k+1} \le \frac{B_{j_{k+2}} - A_{j_{k+1}}}{j_{k+2} - j_{k+1}}. \tag{15}$$

The inequalities (14) and (15) contradict the inequality (13) if the number k is changed by $k+1$. So the inequality $z_{k+1} > z_{k+2}$ is valid and the value z_{k+2} will have the upper estimate

$$z_{k+2} \le \frac{B_{j_{k+2}} - A_{j_{k+1}}}{j_{k+2} - j_{k+1}}. \tag{16}$$

Continuing this process step by step we'll reach the subset $J_{k+m-2,k+m-1}$. Assuming number m (and $m-2$) be even (the case when m be odd can be considered by the same way) we'll have $M_{j_{k+m-2}} = B_{j_{k+m-2}}$, $M_{j_{k+m-1}} = A_{j_{k+m-1}}$, and the following lower estimate for z_{k+m-1}:

$$z_{k+m-1} \ge \frac{A_{j_{k+m-1}} - B_{j_{k+m-3}}}{j_{k+m-1} - j_{k+m-3}}. \tag{17}$$

At the subset $J_{k+m-1,k+m}$ we have $x_i = z_{k+m-1}$ = const, therefore this value can be evaluated in accordance with (7):

$$\frac{A_{j_{k+m}} - B_{j_{k+m-1}}}{j_{k+m} - j_{k+m-1}} \le z_{k+m-1} \le \frac{B_{j_{k+m}} - A_{j_{k+m-1}}}{j_{k+m} - j_{k+m-1}}. \tag{18}$$

Combining estimates (17) and (18) we'll receive:

$$\frac{A_{j_{k+m-1}} - B_{j_{k+m-2}}}{j_{k+m-1} - j_{k+m-2}} \le z_{k+m-1} \le \frac{B_{j_{k+m}} - A_{j_{k+m-1}}}{j_{k+m} - j_{k+m-1}}. \tag{19}$$

It's not difficult to see that the last inequality contradicts the inequality (13) with replacing k by $k+m-1$. So our assumption about a vector having only $(m-1)$ changes at a subset of the length $(m+1)$ is not right. The lemma is proved.

The assertion of the theorem results from the assertion of the lemma, because the set $J_{0,q+1} = \{1,2,\dots n\}$ is a particular case of the set $J_{k-1,k+m}$ with $k = 1$ and $m = q$.

Applications

The theorem 4 gives the lower bound of the number of changes for any feasible vector $X \in D$. A heuristic algorithm to calculate plans having the number of changes equal or close to the lower bound is elaborated (presented below).

Step 1. Initial data: n is the dimension of problem, $A_i < B_i, i = \overline{1, n-1}$ are the lower and upper constraints, $A_n = B_n, C = 0, K = 0$.

Step 2. Calculation $x_i(A) = \frac{A_i}{i}$ and $x_i(B) = \frac{B_i}{i}$, $i = \overline{1, n}$.

Step 3. Comparison: $max x_i(A) \le min x_i(B), i = \overline{1, n}$ if no then go to 5.

Step 4. For $i = \overline{K+1, n}$ the family of solutions has the following form:

$x_i = max x_i(A) + \left(min x_i(B) - max x_i(A)\right)t$, $t \in [0,1]$, END.

Step 5. Find such a number k that satisfies the inequalities

$x(A) = maxx_i(A) \leq x(B) = minx_i(B), i = \overline{1,k},$ but $maxx_i(A) > minx_i(B), i = \overline{1,k+1}.$

Step 6. Calculation $A = kx(A), B = kx(B).$

Step 7. Calculation $y_i(A) = \frac{A_i - A}{i-k},\ y_i(B) = \frac{B_i - A}{i-k}, i = \overline{k+1,n}.$

Step 8. Comparison: $(A) = maxy_i(A) \leq y(B) = miny_i(B), i = \overline{k+1,n}$; if no then go to 11.

Step.9. Comparison: $K = 0,$ if no then go to 25.

Step 10. The family of solutions has the following form:

$x_i = x(A),$ for $i = \overline{1,k},$

$x_i = y(A) + (y(B) - y(A))T,\ T \in [0,1],$ for $i = \overline{k+1,n},$ END.

Step 11. Calculation $z_i(A) = \frac{A_i - B}{i-k},\ z_i(B) = \frac{B_i - B}{i-k},\ i = \overline{k+1,n}.$

Step 12. Comparison: $z(A) = maxz_i(A) \leq z(B) = minz_i(B), i = \overline{k+1,n};$ if no then go to 15.

Step 13. Comparison: $K = 0,$ if no then go to 26.

Step 14. The family of solutions has the following form:

$x_i = x(B),$ for $i = \overline{1,k},$

$x_i = z(A) + (z(B) - z(A))T,\ T \in [0,1],$ for $i = \overline{k+1,n}$, END.

Step 15. Find such a number m that satisfies the inequalities

$y(A) = maxy_i(A) \leq y(B) = miny_i(B), i = \overline{k+1,m},$

but $maxy_i(A) > miny_i(B), i = \overline{k+1,m+1}.$

Step 16. Determine such numbers J1 and J2 that $y_{J1}(A) = y(A)$, $y_{J2}(B) = y(B)$.

Step 17. Find such a number p that satisfies the inequalities

$z(A) = maxz_i(A) \leq z(B) = minz_i(B), i = \overline{k+1,p},$

but $maxz_i(A) > minz_i(B), i = \overline{k+1,p+1}.$

Step 18. Determine such numbers J3 and J4 that $z_{J3}(A) = z(A),\ z_{J4}(B) = z(B)$.

Step 19. Comparison: m < p, if no then go to 21.

Step 20. $x_i = \frac{B-C}{k-K}, i = \overline{K+1,k};\ K = k, C = B, A = B + (p-k)z(A), B = B + (p-k)z(B); k = p,$ go to 7.

Step 21. Comparison: $m = p,$ if no then go to 24.

Step 22. Comparison: J1<J2, if no then go to 24.

Step 23. $x_i = \frac{B-C}{k-K}, i = \overline{K+1,k};\ K = k, C = B, A = B + (m-k)z(A), B = B + (m-k)z(B); k =$

m, go to 7.

Step 24. $x_i = \frac{A-C}{k-K}, i = \overline{K+1,k}$; $K = k, C = A, A = A + (m-k)y(A), B = A + (m-k)y(B); k = m$, go to 7.

Step 25. $x_i = \frac{A-C}{k-K}, i = \overline{K+1,k}$;

$x_i = y(A) + (y(B) - y(A))T$, $T \in [0,1]$, $i = \overline{k+1,n}$, END.

Step 26. $x_i = \frac{B-C}{k-K}, i = \overline{K+1,k}$;

$x_i = z(A) + (z(B) - z(A))T$, $T \in [0,1]$, $i = \overline{k+1,n}$, END.

Example 1: The initial sequence is $P_i = 6,2,2,3,2,5,7,6,11,6,5,10,11,10,12,12,3,3,2,2$, n =20, $A_n = B_n = 120$. The volume of the stock is equal to 9, so the set D is determined by the lower and upper constraints:

A_i: -3,-1,1, 4, 6, 11, 18,24,32,41,46,56,67,77,89,**101**,104,107,109.

B_i: 6, 8,10,13,**15,20,**27,**33**, 44,50,**55**,65,76,86,98,110,113,116,118.

The active constraints are: $B_5 = 15$, $B_6 = 20$, $B_8 = 33$, $B_{11} = 55$, $A_{16} = 101$. So the optimal vector $X^0 =$ (3,3,3,3,3; 5; 6.5,6.5; 22/3,22/3,22/3; 9.2,9.2,9.2,9.2,9.2; 4.75,4.75,4.75,4.75) has 5 changes of its coordinates. We choose the following active constraints: $j_1 = 11$, $B_{11} = 55$; $j_2 = 16, A_{16} = 101$. Designating $j_0 = 0$, $A_0 = 0$; $j_3 = 20$, $B_{20} = 120$; we'll receive:

$$y_1 = \frac{55-0}{11-0} = 5,\ y_2 = \frac{101-55}{16-11} = 9.2,\ y_3 = \frac{120-101}{20-16} = 4.75.$$

All assumptions of the theorem 4 are satisfied, so the lower bound of changes for any vector $X \in D$ is equal to 2. The algorithm presented above gives the family of vectors with 2 changes:

$$x_i = 3, i = \overrightarrow{1,8}; x_i = 9\frac{5}{8} + \frac{19}{72}t, i = \overrightarrow{9,17}; x_i = 3\frac{1}{8} - \frac{19}{24}t, i = \overrightarrow{18,20}, 0 \le t \le 1.$$

Example 2: The initial sequence is $P_i = 4,3,2,5,4,8,7,10,6,6,2,3,6,3,4,6,8,9,8,6,12,14,13,16,10$, n = 25, $A_n = B_n = 175$. The volume of the stock is equal to 5, so the set D is determined by the lower and upper constraints:

A_i: - 1,2,4,9, 13,21,28,**38,**44,**50,**52,55,61,64,68,74,82,91, 99, 105,117,131,144,160.

B_i: 4, 7,**9,**14,**18,**26,33,43,49,55,57,60,66,69,**73**,**79**,87,96,104,**110**,**122,**136,149,165.

The active constraints are: $B_3 = 9$, $B_5 = 18$, $A_8 = 38$, $A_{10} = 50$, $B_{15} = 73$, $B_{16} = 79$, $B_{20} = 110$, $B_{21} = 122$. The optimal vector $X^0 = (3,3,3;\ 4.5,4.5;\ 20/3,20/3,20/3;\ 6,6;\ 4.6,4.6,4.6,4.6,4.6;\ 6;\ 7.75,7.75,7.75,7.75;\ 12;\ 13.25,13.25,13.25,13.25)$ has 8 changes of its coordinates. We take the following active constraints:

$j_1 = 5,\ B_5 = 18,\ j_2 = 10, A_{10} = 50;\ j_3 = 20, B_{20} = 110.$

Designating $j_0 = 0,\ A_0 = 0,\ j_4 = 25, A_{25} = 175,$ we'll receive:

$y_1 = \frac{18-0}{5-0} = 3.6, y_2 = \frac{50-18}{10-5} = 6.4, y_3 = \frac{110-50}{20-10} = 6, y_4 = \frac{175-110}{25-20} = 13.$

All assumptions of the theorem 4 are satisfied, so the lower bound of changes for any vector $X \in D$ is equal to 3. The algorithm presented above gives the family of vectors with 4 changes: $X^* = (3,3,3,3,3;\ 7\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3}, 7\frac{2}{3};\ 2\frac{14}{15}, 2\frac{14}{15}, 2\frac{14}{15}, 2\frac{14}{15}, 2\frac{14}{15};\ 8\frac{1}{6} + \frac{7}{30}t, 8\frac{1}{6} + \frac{7}{30}t, 8\frac{1}{6} + \frac{7}{30}t,\ 8\frac{1}{6} + \frac{7}{30}t, 8\frac{1}{6} + \frac{7}{30}t, 8\frac{1}{6} + \frac{7}{30}t;\ 14.5 - \frac{7}{20}\mathrm{t}, 14.5 - \frac{7}{20}\mathrm{t}, 14.5 - \frac{7}{20}\mathrm{t}, 14.5 - \frac{7}{20}\mathrm{t}), 0 \le t \le 1.$

Conclusions

Two linked problem of optimization subject to two-sided constraints are considered.

For the first problem of optimization of convex separable function of many variables simple necessary and sufficient conditions are received. The theorem 2 gives the necessary and sufficient conditions for the optimal vector to be nonnegative. The theorem 3 gives the possibility to determine in advance all active constraints and implies the decomposition method to find optimal vector. This method allows solving problems with rather big number of variables.

The theorem 4 gives the method to estimate the least number of changes in components' values of any feasible vector. In the particular case when lower and upper active constraints strictly follow one another the optimal vector for the first problem coincides with one of optimal vectors for the second problem. The examples above demonstrate a certain effectiveness of suggested models of rhythmical production and methods of their solution.

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