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Additive Functions from Measure Theory Point-View

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Abstract

We consider the behavior of additive functions from the measurability point of view. Some related questions concerning measurability properties of additive functions with respect to various classes of measures are also discussed.

Keywords: additive function, measurability of function, thick graph.

It is well known that the theory of additive functions plays a significant role in various questions of modern analysis. The measurability of additive functions is an important property for these functions. Moreover, measurability property of additive functions is frequently crucial in the process of investigation of many interesting topics of the theory of differential equations, the theory of equidecomposability of polyhedra,etc. In this connection, see e.g. [1]-[6], [9], [10].

Among functional equations the most famous is Cauchy's classical functional equation

$$f(x+y)=f(x)+f(y).$$

Every solution of Cauchy's equation is trivially an additive function acting from $\mathbf{R}$ into $\mathbf{R}$.

The next statement is valid.

Lemma 1. *If an additive function*

$$f:\mathbf{R}\rightarrow\mathbf{R}$$

satisfied one of the following conditions, then there exists a real constant k *such that*

$$f(x)=k\cdot x$$

for all $x\in\mathbf{R}$ *:*

f is continuous at a point;

f is monotonic on an interval of positive length;

f is bounded from above or below on an interval of positive length;

f is integrable;

f is Lebesgue measurable;

f is a Borel function.

The proof of Lemma 1 can be found [2].

Also, it is well known that every additive function

$$f:\mathbf{R}\to\mathbf{R}$$

which is not of the form

$$f(x)=k\cdot x,$$

for all $x\in\mathbf{R}$, satisfied the following conditions:

(a) f is nonmeasurable with respect to the Lebesgue measure;

(b) the graph of f is dense in $\mathbf{R}^2$.

In the present paper, an approach to some questions of theory of additive functions is discussed, which is rather useful in certain situations and present several statements, which are generalizations of the properties (a) and (b) from a certain point of view.

Throughout this article, we use the following standard notation:

$\mathbf{Q}$ is the set of rational numbers;

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{c}$ is the cardinality of the continuum;

$\mathrm{dom}(\mu)$ is the domain of a given measure μ;

$\mathrm{ran}(f)$ is the range of a given function f;

μ' is the completion of a given measure μ.

Notice that measure of a subset X of an abstract set E is an inner property of X and yields an extremely important real-valued characteristic of X. Unfortunately, not every subset X of an uncountable set E can lie in the domain of a given measure μ and what is more, it turns out in many situations that an arbitrary set Y which is measurable with respect to this μ and has strictly positive μ-measure contains a non-measurable subset with respect to μ. Because of this circumstance it seems natural, instead of the question of investigating the measurability of sets and functions with respect to a concrete measure μ on E, to turn attention to the more general question of investigating the measurability of sets and functions with respect to a given class M of sigma-finite measures on E.

This approach seems to be a natural and helpful generalization of the classical definition of the measurability of real-valued functions and sets with respect to a fixed single measure μ on E. According to this general approach, if M is a given class of σ-finite measures on E, then all real-valued functions f defined on E can be of the following three categories:

absolutely non-measurable functions with respect to M (i.e., all those functions f which are not measurable with respect to every measure from M);

relatively measurable functions with respect to M (i.e., all those functions f for which there exists at least one measure μ from M (certainly, depending on f) such that f turns out to be μ-measurable);

absolutely (or universally) measurable functions with respect to M (i.e., all those functions f which are measurable with respect to any measure from M).

About of this approach be found [9]-[14]. The necessary restriction to σ-finite measures is caused by a number of reasons. One of the main reasons is that one can automatically apply to the class of σ-finite measures the standard theorems of real analysis, such as: Fubinis theorem, Radon-Nikodyms theorem, equivalence between σ-finite and probability measures, etc., which fail to be true in the class of non-σ-finite measures. It should be noticed that the investigation of the question of measurability of functions with respect to certain classes of σ-finite measures quite often needs deep methods from abstract set theory, descriptive set theory, general topology and the theory of groups (for instance, the method of transfinite induction, the Continuum Hypothesis (CH), Martins Axiom (MA), properties of generalized Luzin sets, properties of generalized Sierpinski sets, Bernstein type constructions of some pathological sets, constructions of functions with thick graphs, the techniques of Hamel bases, Kodaira-Kakutanis method of extensions of measures, structural theorems for uncountable commutative groups, etc). These tools are very useful and lead to much deeper understanding the phenomenon of measurability of functions with respect to different classes of -finite measures.

Let μ be a measure on E. As usual, we say that μ is diffused (or continuous) if it vanishes on all singletons in E (i.e., $\mu(\{x\}) = 0$ for each point $x \in E$).

For any set E, let M_E be the class of all nonzero σ-finite diffused measures on E.

Let E be a topological space whose all singletons belong to the Borel σ-algebra of E. We recall that E is a universal measure zero space if there exists no nonzero sigma-finite diffused Borel measure on E. It is well known that there are uncountable universal measure zero subspaces of the real line $\mathbf{R}$. One classical construction of such a subspace of $\mathbf{R}$ is due to Luzin.

Using the notion of a universal measure zero space, a characterization of absolutely nonmeasurable functions (with respect to the class M_E) can be obtained.

Lemma 2. *A function $f : E \to \mathbf{R}$ is absolutely nonmeasurable with respect to M_E if and only if the following two conditions hold:*

(1) for each $x \in \mathbf{R}$, the set $f^{-1}(x)$ is at most countable;

(2) the set $ran(f)$ (i.e. the range of f) is a universal measure zero subspace of $\mathbf{R}$.

The proof of Lemma 2 see, [12].

Assuming some additional set-theoretical axioms, it is not difficult to demonstrate that there exists an absolutely nonmeasurable function

$$f : \mathbf{R} \to \mathbf{R}$$

with respect to the class $M_{\mathbf{R}}$.

Indeed, let L be a generalized Luzin set (see, for example, [9], [10]). It is well known that every generalized Luzin set is a universal measure zero space. In particular, the Luzin set is a universal measure zero subset of $\mathbf{R}$. Let

$$f:\mathbf{R}\to\mathbf{R}$$

be some isomorphism between the two sets $\mathbf{R}$ and L, which both are regarded as vector spaces over $\mathbf{Q}$. Then f can be considered as an injective group homomorphism from $\mathbf{R}$ into $\mathbf{R}$ with $ran(f)=L$ and therefore, f is nontrivial solution of the Cauchy functional equation.

According to the above-mentioned Lemma 2 f turns out to be absolutely nonmeasurable with respect to the class $M_{\mathbf{R}}$.

Theorem 1. *There exists a nontrivial solution of the Cauchy functional equation absolutely nonmeasurable with respect to the class* $M_{\mathbf{R}}$.

In this context, let us remind that any nontrivial solution of the Cauchy functional equation is necessarily nonmeasurable in the Lebesgue sense.

Now, let P be a nonempty linearly independent subset of $\mathbf{R}$ over the field $\mathbf{Q}$ and let $\{e_i, i\in I\}$ be some Hamel basis of $\mathbf{R}$ containing P, where $\mathbf{Q}$ is the set of all rational numbers.

Every real number x admits a unique representation in the form

$$x=q_{i_1}e_{i_1}+q_{i_2}e_{i_2}+\ldots+q_{i_n}e_{i_n},$$

where $n=n(x)$ is a natural number, $i_1,i_2,\ldots i_n$ is a finite injective family of indices from I and $q_{i_1},q_{i_2},\ldots q_{i_n}$ is a finite family of nonzero rational numbers.

We put

$$f(x)=q_{i_1}+q_{i_2}+\ldots+q_{i_n}.$$

It is clear that f is an additive function acting from $\mathbf{R}$ into $\mathbf{Q}$ and the restriction $f\,|\,P$ is identically equal to 1.

Let μ be a Borel diffused probability measure on $\mathbf{R}$ concentrated on P, i. e., $\mu(\mathbf{R}\setminus P)=0$, and let μ' denote the completion of μ.

Now, we can formulate the following statement.

Theorem 2. *There exists an additive function*

$$f : \mathbf{R} \to \mathbf{Q}$$

measurable with respect to the completion μ' *of some nonzero* σ*-finite Borel diffused measure* μ *on* ***R***.

This f *is a nontrivial solution of Cauchy's functional equation (so is nonmeasurable in the Lebesgue sense) and is simultaneously measurable with respect to* μ'.

Consequently, the function f is relatively measurable with respect to the class of all completions of nonzero σ-finite diffused Borel measures $\mathbf{R}$.

Also, it is well known that nontrivial solutions of the Cauchy functional equation are closely connected with equidecomposability theory of polyhedra, in particular, with the so-called Denh invariants of polyhedra (see, [4], [5]).

The next theorem strengthens this result.

Theorem 3. *Among the nontrivial solutions of the Cauchy functional equations one can meet those ones which are absolutely nonmeasurable with respect to the class of all translation invariant measures on the real line* $\mathbf{R}$, *extending the Lebesgue measure.*

Proof. For establishing this fact, consider $\mathbf{R}$ as a vector space over the field $\mathbf{Q}$. Take an arbitrary $e \in \mathbf{Q} \setminus \{0\}$. It is well known that, one-element set e can be extended to a basis of $\mathbf{R}$, that is there exists a Hamel basis $\{e_i : i \in I\}$ for $\mathbf{R}$, containing e. The latter means that $\{e_i : i \in I\}$ is a maximal (with respect to inclusion) linearly independent (over $\mathbf{Q}$) family of elements of $\mathbf{R}$ and $e \in \{e_i : i \in I\}$. Now, find the index $i_0 \in I$ for which $e_{i_0} = e$ and consider the vector subspace V of $\mathbf{R}$ generated by the family $\{e_i : i \in I \setminus \{i_0\}\}$. It is obvious that V turns out to be a vector space in $\mathbf{R}$, complementary to thevector subspace $\mathbf{Q}$. In other words, we have the representation

$$\mathbf{R} = V + \mathbf{Q}, \qquad (V \cap \mathbf{Q} = \{0\})$$

of the space $\mathbf{R}$ in the form of a direct sum of its two vector subspaces. In particular, for each $x \in \mathbf{R}$, the relation

$$\mathrm{card}(\mathrm{V} \cap (\mathrm{x} + \mathbf{Q})) = 1$$

is true, from which it follows that V is a certain Vitali subset of $\mathbf{R}$.

For any $x \in \mathbf{R}$, we have the unique representation

$$x = v + q \qquad v \in V, (q \in \mathbf{Q}).$$

Consider a function

$$f : \mathbf{R} \to \mathbf{R}$$

defined by the formula:

$$f(x) = q \qquad (x \in \mathbf{R})$$

Obviously,

$$f(x + y) = f(x) + f(y) \qquad (x \in \mathbf{R}, y \in \mathbf{R}).$$

We thus conclude that f turns out to be an additive functional on $\mathbf{R}$. Also, an straightforvard verification shows that f is not measurable with respect to every translation invariant measure on the real line $\mathbf{R}$, extending the Lebesgue measure. This follows from the fact, that

$$f^{-1}(0) = V$$

and V is nonmeasurable with respect to every translation invariant measure on the real line $\mathbf{R}$, extending the Lebesgue measure.

This finishes the proof of the Theorem 3.

Let (E_1, S_1, μ_1) and (E_2, S_2, μ_2) be two measurable spaces equipped with sigma-finite measures. We recall that a graph $\Gamma \subset E_1 \times E_2$ is $(\mu_1 \times \mu_2)$-thick in $E_1 \times E_2$ if for each $(\mu_1 \times \mu_2)$-measurable set $Z \subset E_1 \times E_2$ with $(\mu_1 \times \mu_2)(Z) > 0$, we have $\Gamma \cap Z \neq 0$.

In other words, a graph $\Gamma \subset E_1 \times E_2$ is μ-thick in $E_1 \times E_2$ if the equality $\mu_*((E_1 \times E_2) \setminus \Gamma) = 0$, where μ_* denotes the inner measure associated with μ (see, [7], [9], [13]).

The next auxiliary statement is valid.

Lemma 3. *The real line* $\mathbf{R}$ *can be represented in the form*

$$\mathbf{R} = X_1 + X_2 \qquad (X_1 \cap X_2 = \{0\}),$$

where X_1 *and* X_2 *are Bernstein subsets of* $\mathbf{R}$ *and simultaneously, they are vector spaces over the field* $\mathbf{Q}$ *of all rational numbers.*

The proof of Lemma 3 can be found [9].

The next statement holds true.

Theorem 4. *There exists an additive function*

$$f : \mathbf{R} \to \mathbf{R}$$

having the following property: for any σ*-finite diffused Borel measure* μ *on* $\mathbf{R}$ *and for any* σ*-finite measure* ν *on* $\mathbf{R}$*, the graph of* f *is a* $(\mu \times \nu)$*-thick subset of the Euclidean plane* $\mathbf{R}^2$.

Proof Applying Lemma 3 and of the equalities

$$card(X_1) = \mathbf{c}, \; card(\mathbf{R}) = \mathbf{c},$$

the vector spaces X_1 and $\mathbf{R}$ (vector spaces over $\mathbf{Q}$) are isomorphic to each other. Let

$$\phi : X_1 \to \mathbf{R}$$

denote some isomorphism these two spaces.

Let us put

$$f(x) = f(x_1 + f(x_2) = \phi(x_1),$$

where $x_1 \in X_1$ and $x_2 \in X_2$. Obviously, f is an additive function on $\mathbf{R}$. Fix a σ-finite diffused Borel measure μ on $\mathbf{R}$ and a σ-finite measure ν on $\mathbf{R}$.

We assert that the graph of f is $(\mu \times \nu)$-thick subset of the Euclidean plane $\mathbf{R}^2$. Indeed, If Z is an arbitrary $(\mu \times \nu)$-measurable set with $(\mu \times \nu)(Z) > 0,$ then there exists a point $t \in \mathbf{R}$ such that $\mu(Z(t)) > 0$, where

$$Z(t) = \{x \in \mathbf{R} : (x,t) \in Z\}.$$

Consider the point

$$x_1 = \phi^{-1}(t) \in X_1.$$

Since X_2 is Bernstein subset of $\mathbf{R}$, we have

$$X_2 \cap (Z(t) - x_1) \neq \varnothing.$$

Choose a point x_2 from the $X_2 \cap (Z(t) - x_1)$ and define

$$x = x_1 + x_2.$$

Then we get

$$x = x_1 + x_2 \in Z(t), \qquad (x,t) \in Z,$$

$$(x, f(x)) = (x,t), \qquad t = f(x), (x, f(x)) \in Z,$$

which completes the proof of Theorem 4.

Notice that, the thickness of graphs is pathological phenomenon for subsets of $\mathbf{R}^2$. However, this feature plays an essential role in the problem of extensions of measures (see, [7]-[10], [13]). Indeed, fix a Borel diffused probability measure ν on $\mathbf{R}$. Now, let μ be any sigma-finite diffused Borel measure on $\mathbf{R}$. For each $(\mu_1 \times \nu)$-measurable set $Z \subset \mathbf{R}^2$, we denote

$$Z' = \{x \in \mathbf{R} : (x, f(x)) \in Z\}.$$

Further, we put

$$S = \{Z' : Z \in dom(\mu \times \nu)\}.$$

It can easily be verified that S is a sigma-algebra of subsets of $\mathbf{R}$. We define a functional μ_1 on S by the formula

$$\mu_1(Z') = (\mu \times \nu)(Z) \qquad (Z \in dom(\mu \times \nu)).$$

It is easy to show that the definition of μ_1 is correct in view of the $(\mu \times \nu)$-thickness of the graph of f. Also, μ_1 turns out to be a measure on S, which extends the original measure μ.

We thus come to the following statement.

Theorem 5. *The additive function*

$$f:\mathbf{R}\to\mathbf{R}$$

is relatively measurable with respect to the class of all extensions of σ-finite diffused Borel measures on $\mathbf{R}$.

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