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Performance of T^2 Chart over $\bar{x}$ Chart for Monitoring the Process Mean: A Simulation Study

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Abstract

In real life we observe that there are many situations in which the simultaneous monitoring or control of two or more related quality characteristics is necessary. In such situations we can use univariate control charts to each individual quality characteristics but these control charts can lead to erroneous conclusions. Multivariate methods that consider the quality characteristics jointly are required. The most familiar multivariate process monitoring and control procedure is the Hotelling T^2 control chart for monitoring the process mean simultaneously. It is a direct analog of the univariate Shewhart $\bar{x}$ control chart. In this paper, using simulation study we show that if the quality characteristics are related then for monitoring the mean of the process Hotelling T^2 control chart performs better than Shewhart $\bar{x}$ control chart.

Keywords: Quality characteristics; Shewhart control chart; multivariate control chart; detection rate of out of control signal.

1. Introduction

Statistical process control (SPC) is a fundamental methodology consisting of many techniques that have been proven useful in quality and productivity improvement of products and processes. Among these techniques, the control chart is the featured technique for keeping processes in control by monitoring key quality characteristics of interest. Univariate control charts have been devised to monitor a number of process variable, where as a multivariate control charts monitor a number of process variables simultaneously. The $\bar{x}$ chart is well known Shewhart control chart for monitoring process mean and useful for monitoring only one characteristic. The most popular multivariate control charts is named by

Hotelling T^2 control chart. In practice many process monitoring and control scenarios involve several related variables. Although applying univariate control charts to each individual variable is a possible solution, we will see that this is inefficient and can lead to erroneous conclusions. Multivariate methods consider the variables jointly are required in this situation.

Statistical methods and their application in quality improvement have a long history. Shewhart (1924) of the Bell Telephone Laboratories developed the statistical control chart concept, which is often considered the formal beginning of statistical quality control (Montgomery, 2007). The first original study in multivariate quality control was introduced by Hotelling (1947). Before this study, he wrote a paper on T^2 test procedure for multivariate population in 1931. Hotelling (1947) applied his procedures to bomb sight data during World War II. Subsequent papers dealing with control procedures for several related variables include Hicks (1955), Jackson (1956, 1959), Edward Jackson (1985), Crosier (1988), Hawkins (1991, 1993), Lowry and Montgomery (1995), Pignatiello and Runger (1990), Tracy (1992), Montgomery and Wadsworth (1972). A comparison of the univariate out of control signals with the multivariate out of control signals is made by El-Din et al. (2006) to illustrate the efficiency of the Hotelling T^2. Also Haridy and Wu (2009) compare univariate and multivariate quality control charts for monitoring dynamic behavior processes.

In industry, there are several reasons for shifting the mean of a process. This mean shifting of the process indicates an out of control condition of the process and increases the production cost, or defects item. To reduce the production cost or defect items it is very important to identify the mean shifting of the process. A control chart showing out of control signal is used to identify shifting of the process mean. In this paper, using simulation study we are interested to compare the probability of detection (detection rate) of out of control signal for different mean shift in Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart.

2. Methodology

If a product is to meet or exceed customer expectations, generally it should be produced by a process that is stable or repeatable. More precisely, the process must be capable of operating with little variability around the target or nominal dimensions ofthe product’s quality characteristics. Statistical process control (SPC) is a powerful collection of problem-solving tools useful in achieving process stability and improving capability through the reduction of variability.

2.1. A Typical Control Chart

A typical control chart is shown in Figure 1.

The control chart is a graphical display of a quality characteristic that has been measured or computed from a sample versus the sample number or time. The chart contains a center line that represents the average value of the quality characteristic corresponding to the in-control state. (That is, only chance causes are present.) Two other horizontal lines, called the upper control limit (UCL) and the lower control limit (LCL), are also shown on the chart. These control limits are chosen so that if the process is in control, nearly all of the sample points will fall between them. As long as the points plot within the control limits, the process is assumed to be in control, and no action is necessary. However, a point that plots outside of the control limits is interpreted as evidence that the process is out of control, and investigation and corrective action are required to find and eliminate the assignable cause or causes responsible for this behavior.

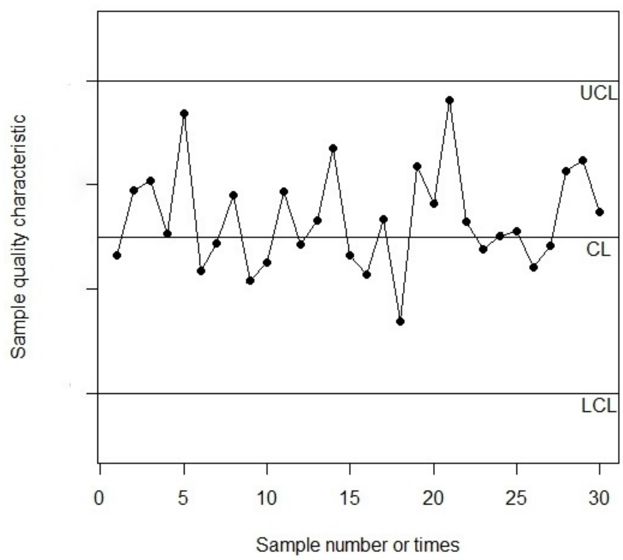

Figure 1. A typical control chart

2.2. Shewhart $\overline{\boldsymbol{x}}$ *Control Chart*

The x-bar chart is used for monitoring the mean of a characteristic in statistical quality control or is used to detect a change in the level of a process.

Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ, where both μ and σ are known. If x_1, x_2, . . ., x_n is a sample of size n, then the average of this sample is

$$\bar{x} = \frac{x_1 + x_2 + \ldots + x_n}{n},$$

and we know that $\bar{x}$ is normally distributed with mean μ and standard deviation $\sigma_{\bar{x}}$. Furthermore, the probability is 1-α that any sample mean will fall between

$$\mu \pm Z_{\alpha/2}\sigma_{\bar{x}} = \mu \pm Z_{\alpha/2}\frac{\sigma}{\sqrt{n}}.$$

Therefore, if μ and σ are known, the equation could be used as upper and lower control limits on a control chart for sample means. It is customary to replace $Z_{\alpha/2}$ by 3, so that three-sigma limits are employed. If a sample mean falls outside of these limits, it is an indication that the process mean is no longer equal to μ.

We have assumed that the distribution of the quality characteristic is normal. However, the above results are still approximately correct even if the underlying distribution is non-normal, because of the central limit theorem.

In practice, we do not know μ and σ. Therefore, they must be estimated from preliminary samples or subgroups taken when the process is thought to be in control. These estimates should usually be based on at least 20 to 25 samples. Suppose that m samples are available, each containing n observations on the quality characteristic. Typically, n will be small, often either 4, 5, or 6. These small sample sizes usually result from the construction of rational subgroups and from the fact that the sampling and inspection costs associated with variables measurements are usually relatively large. Let $\bar{x}_1$, $\bar{x}_2$, . . ., $\bar{x}_m$ be the average of each sample. Then the best estimator of μ, the process average, is the grand average-say,

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \ldots + \bar{x}_m}{m}.$$

Thus $\bar{\bar{x}}$ would be used as the center line on the $\bar{x}$ chart. To construct the control limits, we need an estimate of the standard deviation σ. If σ^2 is the unknown variance of a probability distribution, then an unbiased estimator of σ^2 is the sample variance

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}.$$

Suppose that m preliminary samples are available, each of size *n*, and let s_i be the standard deviation of the i th sample. The average of the m standard deviations is

$$\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i.$$

Then the control limit for x-bar chart with $CL = \bar{\bar{x}}$ is

$$UCL = \bar{\bar{x}} + A_3\bar{s}$$

$$LCL = \bar{\bar{x}} - A_3\bar{s}, \tag{1}$$

where the quantity A_3 is a statistically determined constant that depends on *n*, which is obtained from statistical table. The term UCL, CL, and LCL is the upper control limit, center line, and lower control limit respectively.

If any subgroup mean is outside of control limits (either LCL or UCL), then the process is statistically out-of-control.

3. Example of Shewhart $\bar{x}$ Control Chart

The "tensile strength" and "diameter" of a textile fiber are two important quality characteristics. Say, the quality engineer decided to use sample size, $n = 10$ fiber specimens in each sample. He has taken 20 preliminary samples ($m = 20$), and on the basis of these data he concludes that $\bar{\bar{x}}_1 = 115.59$ psi, $\bar{\bar{x}}_2$ =

$1.06(\times 10^{-2})$ inch, $\bar{s}_1^2 = 1.23$, and $\bar{s}_2^2 = 0.83$, where $\bar{\bar{x}}_1$ = mean of tensile strength data, $\bar{s}_1^2$ = variance of tensile strength data, $\bar{\bar{x}}_2$ = mean of diameter data and $\bar{s}_2^2$ = variance of diameter data.

The control limits calculated for quality characteristics “tensile strength" is

$$UCL = \bar{\bar{x}}_1 + A_3 \bar{s}_1 = 116.74$$

$$CL = \bar{\bar{x}}_1 = 115.59$$

$$LCL = \bar{\bar{x}}_1 - A_3 \bar{s}_1 = 114.43.$$

Let m_1 = subgroup size of original data and m_2 = additional subgroup size of data. Consider 10 subgroups (m_2=10) for the tensile strength data, and diameter data with 1.0 mean shift, and 0.5 mean shift respectively. We add them to the original subgroups (m_1=20). The Shewhart $\bar{x}$ chart for tensile strength data (original, and mean shifted) is shown in Figure 2:

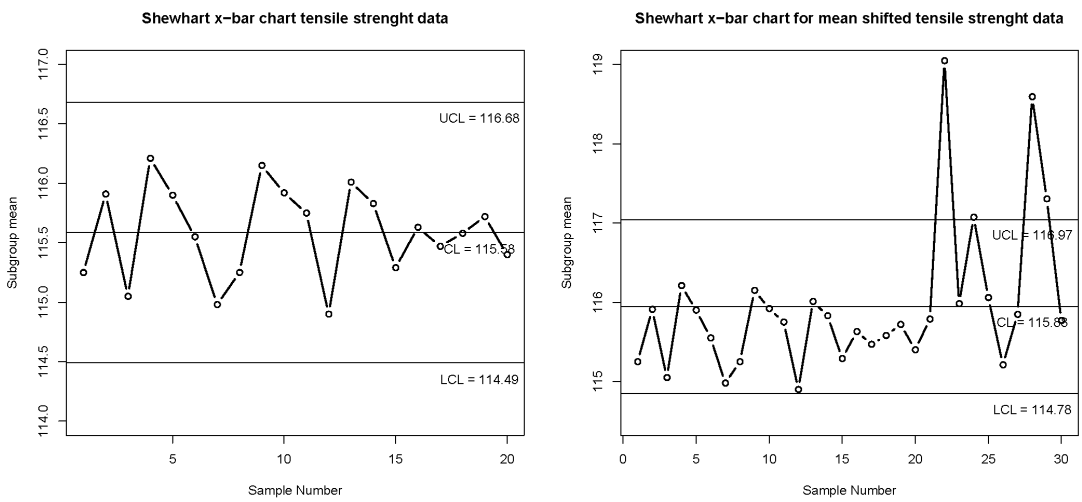

Figure 2. Shewhart x-bar chart for original tensile strength data and mean shifted data

From 1*st* figure of Figure 2 we observe that no points exceeds the control limits, so the Shewhart $\bar{x}$ chart for original quality characteristic “tensile strength" shows that the process is statistically in-control. For 2*nd* figure, we observe that four points outside the control limits indicates for mean shifted quality characteristic the process is statistically out of control.

Similarly we get the following figure for diameter data:

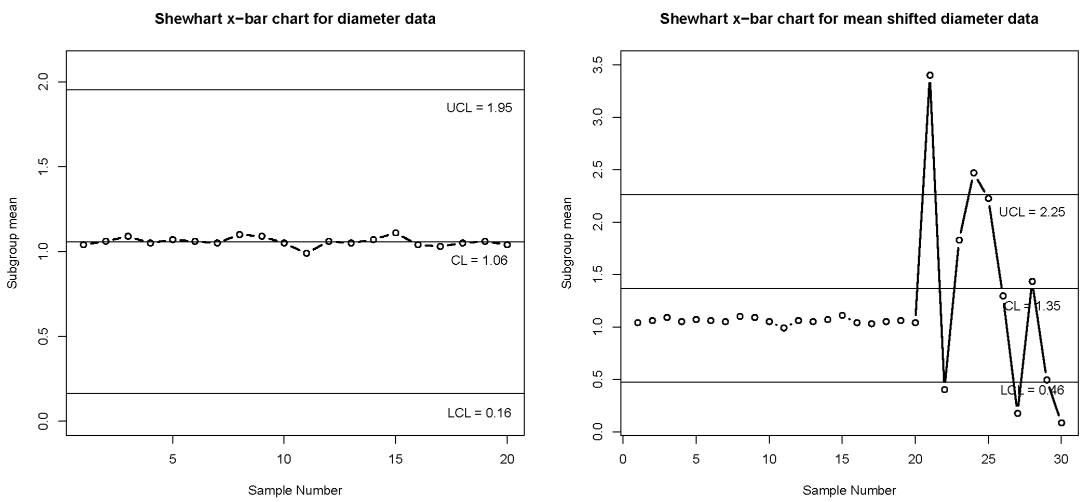

Figure 3. Shewhart x-bar chart for original diameter data and mean shifted data

The interpretation is similar as before. From 1*st* figure of Figure 3 we observe that no points exceeds the control limits, so the Shewhart $\bar{x}$ chart for original quality characteristic “diameter" shows that the process is statistically in-control. For 2*nd* figure, we observe that five points outside the control limits indicates for mean shifted data, the process is statistically out of control.

4. Multivariate Hotelling T^2 Control Chart

The most familiar multivariate process-monitoring and control procedure is the Hotelling T^2 control chart. It is a direct analog method of the univariate Shewhart chart. We present only Hotelling T^2 chart for sub-grouped data.

Let $X_1, X_2, \ldots, X_p$ are p quality characteristics that are correlated. As defined before let n = sample size of each subgroup, and m = subgroup size. Also let $\bar{\boldsymbol{x}}$, $\bar{\bar{\boldsymbol{x}}}$ and $\boldsymbol{S}$ are the subgroup mean vector, overall sample mean vector and sample variance-covariance matrix respectively. Then the Hotelling T^2 statistic is

$$\boldsymbol{T}^2 = n(\bar{\boldsymbol{x}} - \bar{\bar{\boldsymbol{x}}})'\boldsymbol{S}^{-1}(\bar{\boldsymbol{x}} - \bar{\bar{\boldsymbol{x}}}). \tag{2}$$

Hotelling T^2 control chart represent the T^2 values on y-axis and sample number in x-axis. The control limits for the T^2 control chart are given by

$$UCL = \frac{p(m-1)(n-1)}{mn-m-p+1} F_{\alpha,p,mn-m-p+1}$$
$$LCL = 0. \tag{3}$$

When all points fall within the control limit, then the process is statistically in control. If any points fall outside the control limits (either LCL or UCL), then the process is statistically out of control.

5. Example of Hotelling T^2 Control Chart

In the example of Shewhart x-bar chart the covariance of two quality characteristics tensile strength and diameter is $\bar{s}_{12} = 0.79$. For $\alpha = .001$

$$UCL = 13.72,$$

where $n = 10$, m (or m_1) = 20, $m_2 = 10$ & p = number of quality characteristics = 2.

The Hotelling T^2 chart for the tensile strength data and diameter data (original and mean shifted) is shown in Figure 4. From Figure 4, for 1*st* figure we observe that no points exceeds the control limits, so for original data, we would conclude that the process is statistically in control. But for 2*nd* figure there are a lot of points exceed the upper control limit indicates that for mean shifted data, the process is statistically out-of-control.

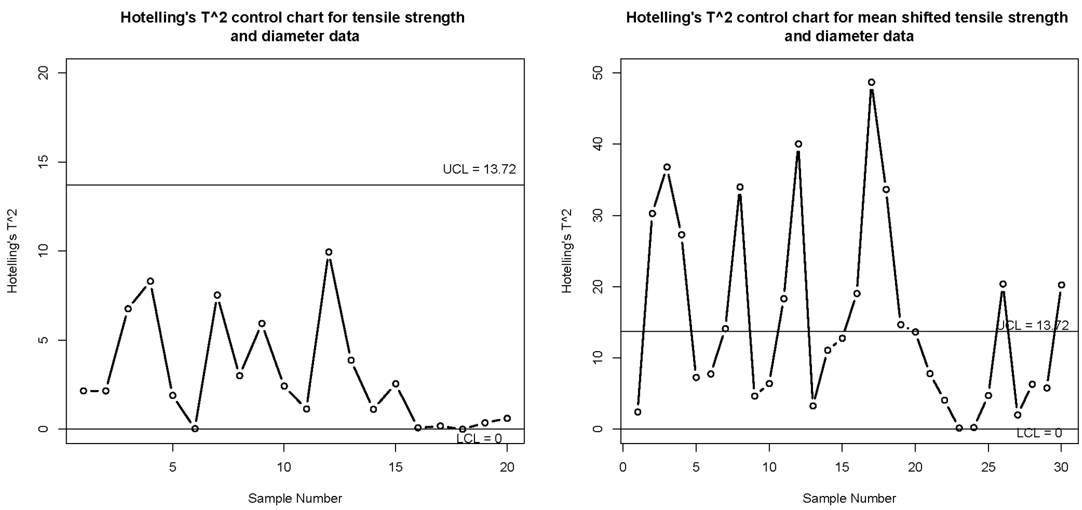

Figure 4. Hotelling T^2 control chart for original tensile strength and diameter data and mean shifted data

6. Detection Rate of Out of Control Signal

The detection rate of out of control signal for any control chart is nothing but a proportion of out of control signal of a process and defined as

$$\text{Detection rate} = \frac{\text{Number of times the process shows out of control}}{\text{Total number of times the process compiled}}. \tag{4}$$

Consider a mean shifted process variable, out of 100 times the process shows 60 times at least one point outside the control limit. Then the detection rate of out-of-control signal is

$$\text{Detection rate} = \frac{60}{100} = 0.60.$$

3. Simulation Study and Results

In this section we show the performance of Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart for correlated bivariate and multivariate (we consider three variables) quality characteristics to control the process mean. We use different shifts in process mean and apply Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart simultaneously.

3.1. Bivariate Case

First we set two correlated bivariate normal quality characteristics, say, x_1 and x_2 with sample mean vector and covariance matrix respectively, $\bar{\bar{\boldsymbol{x}}} = (115, 50)'$ and

$$\boldsymbol{S} = \begin{bmatrix} 1.00 & 0.50 \\ 0.50 & 1.00 \end{bmatrix}.$$

Using this mean vector and covariance matrix we generate 10 subgroups (m_1) each with containing 10 observations, i.e., the sample size (n) for each subgroup is 10.

Now we generate 10 subgroups (m_2) each with containing 10 observations considering same covariance matrix but shifting the mean vector as 0.5σ. That is the mean vector is $\bar{\bar{x}} = (115.5, 50.5)'$ and covariance matrix is

$$\boldsymbol{S} = \begin{bmatrix} 1.00 & 0.50 \\ 0.50 & 1.00 \end{bmatrix}.$$

We now add this 10 subgroups (m_1) with the original 10 subgroup (m_2) and apply Shewhart $\bar{x}$ control chart on these 20 subgroups ($m = m_1 + m_2$) for x_1 and x_2 quality characteristics separately. We repeat this procedure 1000 times and then calculate probability of detection (detection rate) of out of control signal which is shown in Table 1. From this table we observe that the probability of detection (detection rate) is zero for both quality characteristics. This means that the Shewhart $\bar{x}$ control chart for x_1 and x_2 quality characteristics is in control though the process is actually out of control. This indicates that for correlated bivariate quality characteristics, using individual Shewhart $\bar{x}$ control chart leads to wrong conclusion. We now apply Hotelling T^2 control chart on these 20 subgroups ($m = m_1 + m_2$) for x_1 and x_2 quality characteristics jointly and calculate probability of detection of out of control. From Table 1 we observe that the probability is 1 that means Hotelling T^2 control chart accurately identifies that the process is out of control.

Table 1. Detection rate & subgroup effect of $\bar{x}$ chart & T^2 chart simultaneously (bivariate case)

Shift of the mean	Control Chart Name	$m = 20$	$m = 40$	$m = 60$	$m = 100$
0.5σ	Shewhart $\bar{x}_1$	0.04	0.11	0.17	0.26
	Shewhart $\bar{x}_2$	0.05	0.13	0.23	0.31
	Hotelling T^2	1.00			
1σ	Shewhart $\bar{x}_1$	0.39	0.50	0.59	0.63
	Shewhart $\bar{x}_2$	0.41	0.53	0.59	0.67
	Hotelling T^2	1.00			
1.5σ	Shewhart $\bar{x}_1$	0.83	0.91	0.97	1.00
	Shewhart $\bar{x}_2$	0.83	0.94	0.98	1.00
	Hotelling T^2	1.00			
2σ	Shewhart $\bar{x}_1$	0.98	1.00	1.00	1.00
	Shewhart $\bar{x}_2$	1.00	1.00	1.00	1.00
	Hotelling T^2	1.00			

Note: Shewhart $\bar{x}_1$ indicates Shewhart $\bar{x}$ control chart for x_1 variable and Shewhart $\bar{x}_2$ indicates Shewhart $\bar{x}$ control chart for x_2 variable.

Using the same procedure described above keeping the same covariance matrix and just shifting the mean vector as 1σ we generate 10 subgroups and add this subgroup with the original subgroups. The results (Table 1) of individual Shewhart $\bar{x}$ control chart indicates a better result than previous one where mean shift was 0.5σ. Continuing the same procedure we generate the 10 subgroups just shifting the mean as 1.5σ, 2σ and add these 10 subgroups with the original 10 subgroups we can see the effect of Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart. The results of these generations are shown in Table 1. From Table 1 we observe that if we increase the mean shift the probability of detection of out of control increases for Shewhart $\bar{x}$ control chart. This indicates that Shewhart $\bar{x}$ control chart works well for large shift but it leads to wrong conclusion for small shift. Whereas Hotelling T^2 control chart is not sensitive for mean shift, it always shows correct result.

Instead of 10 subgroups we repeat this procedure for 20, 30 and 50 generated original subgroups and add another 20, 30 and 50 respective subgroups generated using the mean shift of the process as 0.5σ, 1σ, 1.5σ, 2σ. The result is shown in Table 1. Table 1 indicates that there is an effect of subgroup size in Shewhart $\bar{x}$ control chart. The probability of detection increases for increasing subgroup size. On the otherhand Hotelling T^2 control chart is not sensitive for subgroup size.

Figure 5 is more helpful to visualize the subgroup size effect in Shewhart x-bar chart and Hotelling T^2 chart.

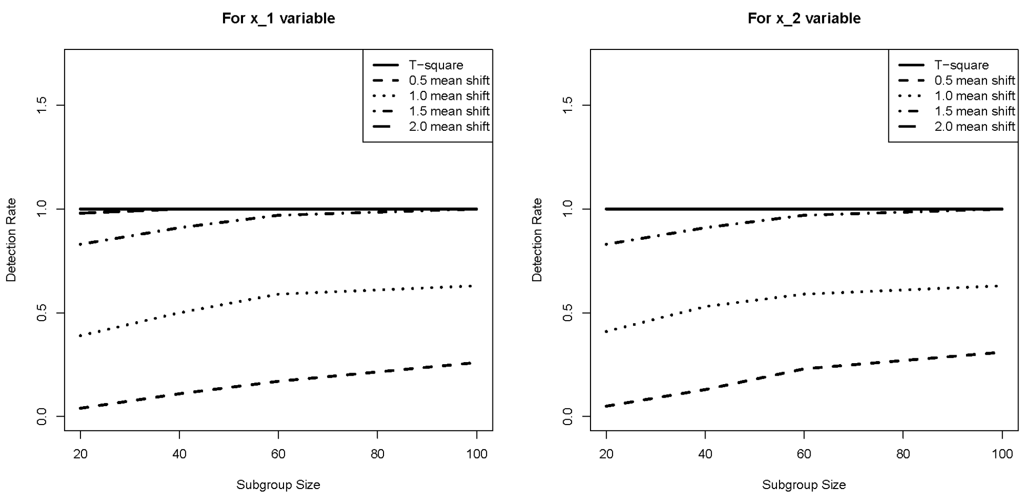

Figure 5. Detection rate & subgroup size effect of $\bar{\boldsymbol{x}}$ chart and T^2 chart (bivariate case)

The detection rate of out of control signal is shown in Figure 5. From this Figure we observe that if we increase the subgroup size the probability of detection is also increases. We also observe that for large shift both the Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart show the same detection rate but for small shift T^2 chart performs better than Shewhart chart.

3. 2. Multivariate Case

Second, to observe the situation of multivariate case, we set three correlated trivariate normal quality characteristics, say, x_1, x_2 and x_3 with sample mean vector and covariance matrix respectively, $\bar{\bar{\boldsymbol{x}}} = (10, 290, 50)'$ and

$$\boldsymbol{S} = \begin{bmatrix} 1.00 & -0.41 & 0.32 \\ -0.41 & 1.00 & 0.54 \\ 0.32 & 0.54 & 1.00 \end{bmatrix}.$$

Then we apply same procedure as in bivariate case. The result is shown in Table 2.

Table 2. Detection rate & subgroup effect of $\bar{\boldsymbol{x}}$ chart and T^2 chart simultaneously (multivariate case)

Shift of the mean	Control Chart Name	$m=20$	$m=40$	$m=60$	$m=100$
0.5σ	Shewhart $\bar{x}_1$	0.05	0.08	0.14	0.17
	Shewhart $\bar{x}_2$	0.07	0.09	0.13	0.17
	Shewhart $\bar{x}_3$	0.07	0.11	0.14	0.16
	Hotelling T^2	1.00			
1σ	Shewhart $\bar{x}_1$	0.49	0.59	0.67	0.73
	Shewhart $\bar{x}_2$	0.43	0.54	0.61	0.67
	Shewhart $\bar{x}_3$	0.51	0.60	0.65	0.68
	Hotelling T^2	1.00			
1.5σ	Shewhart $\bar{x}_1$	0.93	0.98	1.00	1.00
	Shewhart $\bar{x}_2$	0.95	0.97	1.00	1.00
	Shewhart $\bar{x}_3$	0.95	0.98	1.00	1.00
	Hotelling T^2	1.00			
2σ	Shewhart $\bar{x}_1$	1.00	1.00	1.00	1.00
	Shewhart $\bar{x}_2$	1.00	1.00	1.00	1.00
	Shewhart $\bar{x}_3$	1.00	1.00	1.00	1.00
	Hotelling T^2	1.00			

From Table 2 we observe that for small mean shift of a process the Shewhart $\bar{x}$ control chart does not identifies the out of control signal correctly and gives us wrong result. But the probability is 1.0 for detecting out of control by Hotelling T^2 control chart that means it accurately identifies that the process is out of control. Table 2 also indicates that there is an effect of subgroup size in Shewhart $\bar{x}$ control chart. The probability of detection increases for increasing subgroup size. On the other hand Hotelling T^2 control chart is not sensitive for subgroup size.

Figure 6 is more helpful to visualize the detection rates and subgroup size effect in both charts.

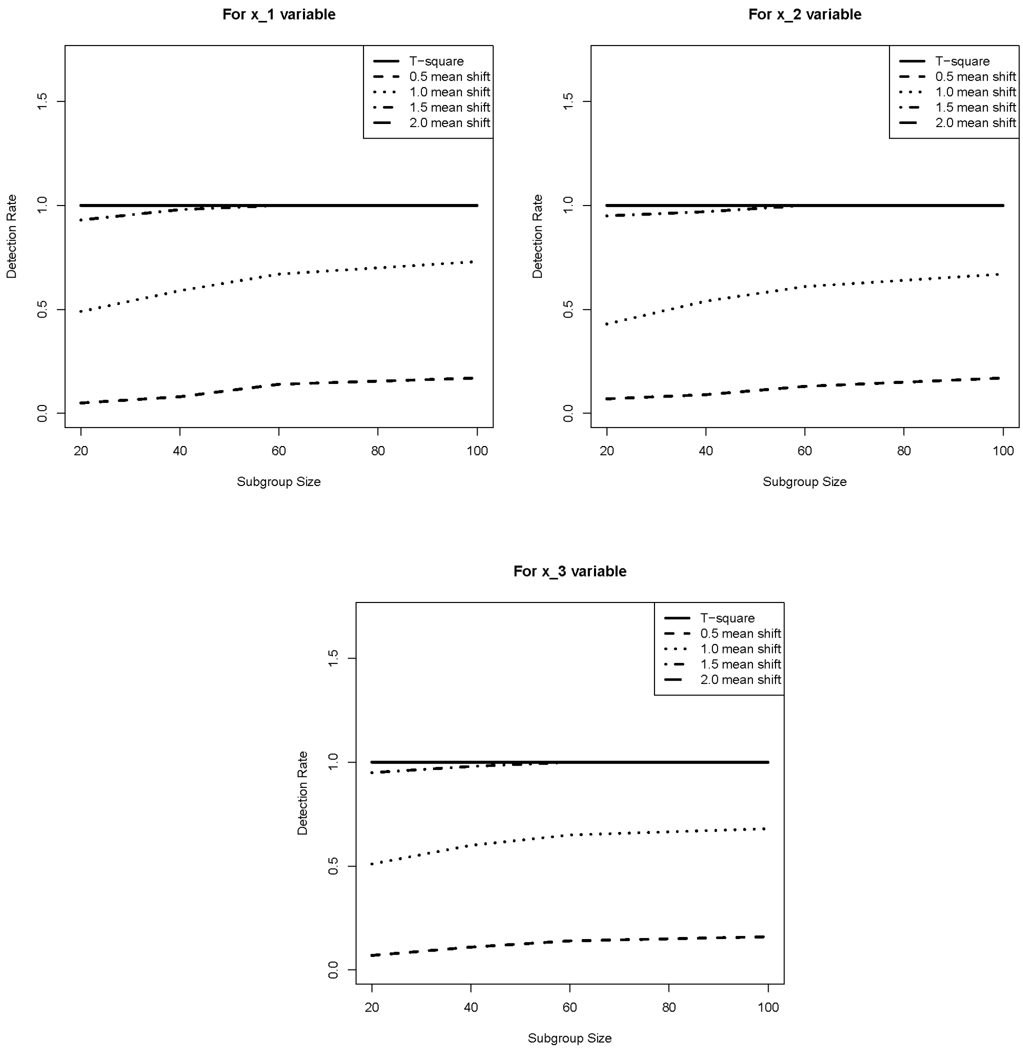

Figure 6. Detection rate & subgroup size effect of $\bar{\mathbf{x}}$ chart and T^2 chart (multivariate case)

The detection rate of out of control signal is shown in Figure 6. Here, the interpretation is same as in bivariate case. From this Figure we observe that if we increase the subgroup size the probability of detection is also increases. We also observe that for large shift both the Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart show the same detection rate but for small shift T^2 chart performs better than Shewhart chart.

4. Conclusion

In industry there are many situations where the quality characteristics are related. To control the process mean of these quality characteristics we use either Shewhart $\bar{x}$ control chart or Hotelling T^2 control chart. In this study using simulation we show the performance of Shewhart $\bar{x}$ control chart and Hotelling T^2 control chart in detecting out of control signal condition of the related quality characteristics.

First we use two related quality characteristics and calculate the detection rate of out of control signal for different shifts in process mean. We observe that for small shift in process mean for example 0.5σ, the Shewhart $\bar{x}$ control chart leads to wrong conclusion. If we increase the shift of the process mean from 1σ to 2σ we observe that the sensitivity of Shewhart $\bar{x}$ control chart increases. For large shift that is 1.5σ or large Shewhart $\bar{x}$ control chart gives the correct result. If we use Hotelling T^2 control chart then this chart shows the correct result for any shift in process mean. We also observe that there is an effect of subgroup size in Shewhart $\bar{x}$ control chart. If we increase the subgroup size then detection rate of out of control signal increases. But we observe that Hotelling T^2 control chart is not sensitive for subgroup size. For trivariate case we observe similar result as bivariate case but here Shewhart $\bar{x}$ control chart shows some improved performance. These simulations suggest that for any number of related quality characteristics Hotelling T^2 control chart should be used. This chart is not sensitive for any shift in process mean and for any number of subgroup sizes.

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