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Reinitiated Laplace Homotopy Ananlysis Method For Solving Integral Equations

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Abstract

The complexity of the deformation equation increases exponentially with the order of approximation. Consequently, implementing the Laplace homotopy analysis method (LHAM) under high deformation order can be very computationally costly and lengthy and even cause computational paralysis in cases. Here, the LHAM is modified in a reinitiated manner where the low order results are initiated for further approximation using truncated Maclaurin expansions. This modified approach manages to avoid high order approximation but still promises accurate approximate series solution. This approach greatly improves the efficiency of LHAM in solving integral equations.

Keywords: Laplace transform, homotopy analysis method (HAM), integral equations.

Introduction

Integral equations (IEs) arise commonly in various modeling of initial value problems appear in physics, chemistry, biology and engineering applications [1, 2]. IEs have been studied and solved by many researchers using various methods, such as, series solution method, Adomian decomposition method, homotopy perturbation method and variational iteration method [3-5]. IEs are also solved using homotopy analysis method (HAM) and Laplace homotopy analysis method (LHAM) [6-8]. To improve the homotopy-approximation accuracy, a typical move is to increase the deformation order. However, as the deformation order goes higher, the number of terms in the deformation equation increases very rapidly and

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hence complicating the deformation equation greatly. This leads to very lengthy and complicated symbolic computation. Hence, the exponential growth of computation time has become a major drawback for LHAM under high deformation order.

To address the proliferation of terms and the computation tediousness, we modify the iterative approach in [9] with the help of the truncated Maclaurin series. The approach keeps low homotopy deformation order but reinitiates the truncated low order homotopy-approximation and iterates the process. This modified approach is named reinitiated LHAM and will be abbreviated as RE-LHAM henceforth. In the following two test examples, the IEs are solved by the standard LHAM and the RE-LHAM respectively to compare and to illustrate the strength of RE-LHAM.

LHAM For Integral Equations

The LHAM starts by performing Laplace transform onto the problem under study, followed by implementing HAM onto the 'Laplace transformed' problem. For illustration, consider a nonlinear IE with the following general form

$$y(t) - f(t) - \int_{a(t)}^{b(t)} K\big(t, u, y(u)\big)\, du = 0, \tag{1}$$

where K is a nonlinear kernel, $f(t)$ is an arbitrary function and $y(t)$ is the unknown function of t to be solved. Taking the Laplace transform onto (1), we have

$$Lap\{y(t)\} - Lap\{f(t)\} - Lap\left\{\int_{a(t)}^{b(t)} K\big(t, u, y(u)\big)\, du\right\} = 0, \tag{2}$$

where Lap denotes the Laplace transform operator. Now, the standard HAM is implemented onto (2) adopting the ideas in [9, 10]. Based on the Laplace transformed problem (2), we define the LHAM nonlinear operator

$$\widetilde{N}\{\varphi(t,q)\} = Lap\{\varphi(t,q)\} - Lap\{f(t)\} - Lap\left\{\int_{a(t)}^{b(t)} K\big(t, q, u, \varphi(u,q)\big)\, du\right\}, \tag{3}$$

where $\varphi(t,q)$ is a function of t and the homotopy-parameter q.

We then construct the zeroth-order deformation equation

$$(1-q)Lap\{\varphi(t,q)-v_0(t)\} = qh\widetilde{N}\{\varphi(t,q)\}, \tag{4}$$

where $q \in [0,1]$ is the homotopy-parameter, h is the convergence-control parameter, and $v_0(t)$ is the initial guess function. This will build a continuous deformation from the initial guess $v_0(t)$ to the true solution of the problem $y(t)$. When $q=0$, we have $\varphi(t,0)=v_0(t)$; whereas when $q=1$, we have $\widetilde{N}\{\varphi(t,1)\}=0$ or equivalently $\varphi(t,1)=y(t)$. Thus, $\varphi(t,q)$ represents a continuous deformation from the initial guess $v_0(t)$ to the solution of the problem $y(t)$ as q increases from 0 to 1.

By Taylor's theorem, we may expand $\varphi(t,q)$ into Maclaurin series with respect to q

$$\varphi(t,q) = v_0(t) + \sum_{j=1}^{\infty} v_j(t)q^j, \tag{5}$$

where $y(0)=v_0$ and

$$v_j(t) = \frac{1}{j!}\frac{\partial^j \varphi(t,q)}{\partial q^j}\bigg|_{q=0} = \widetilde{D}_j\{\varphi(t,q)\}. \tag{6}$$

Here, $\widetilde{D}_j\{\varphi(t,q)\}$ is known as the *j*th-order homotopy-derivative of φ while j is an integer and $j \geq 0$ [10]. Further, $\widetilde{D}_j\{\ \}$ is called the *j*th-order homotopy-derivative operator.

If $h, v_0(t)$ are properly chosen, the series (5) will converges to $y(t)$ at $q=1$ such that we get the LHAM series solution of $y(t)$

$$\varphi(t,1) = y(t) = v_0(t) + \sum_{j=1}^{\infty} v_j(t). \tag{7}$$

In practice, the upper limit of the summation is truncated to a finite integer. So, the *n*th-order LHAM series approximation of $y(t)$ is defined by

$$y(t) \approx v_0(t) + \sum_{j=1}^{n} v_j(t). \tag{8}$$

Applying $\widetilde{D}_j\{\ \}$ onto both sides of (4), we can derive the *j*th order deformation equation

$$\begin{cases} Lap\{v_j(t)\} = h\widetilde{N}\{\varphi(t,0)\} = h\widetilde{N}\{v_0(t)\}, & j=1 \\ Lap\{v_j(t)-v_{j-1}(t)\} = h\widetilde{D}_{j-1}\{\widetilde{N}[\varphi(t,q)]\}, & j\geq 2 \end{cases} \tag{9}$$

To get the LHAM series solution, we need to compute all the $v_j(t)$ for $j \geq 1$ as in (7). In fact, all these v_j's which constitutes the LHAM series solution can be obtained by solving the *j*th order deformation equation (9) for $v_j(t)$. Doing so, one ends up

$$v_j(t) = \begin{cases} h\, Lap^{-1}\{\widetilde{N}[v_0(t)]\}, & j = 1 \\ h\, Lap^{-1}\left\{\widetilde{D}_{j-1}\{\widetilde{N}[\varphi(t,q)]\}\right\} + v_{j-1}(t), & j \geq 2 \end{cases} \tag{10}$$

where $Lap^{-1}$is the inverse Laplace transform operator. Note that this expression gives a recursion relation between $v_j(t)$ and $v_{j-1}(t)$ making it possible to compute all the subsequent $v_j(t)$ when $v_0(t)$ is available. And this is achievable since $v_0(t)$ is the initial guess of our choice.

The Proposed Re-LHAM Approach

In the frame of LHAM (and HAM), one can choose the initial guess $v_0(t)$ freely. However, a good initial guess i.e. one that is closer to the solution, can accelerate the convergence of the homotopy-approximation and logically, the better the initial guess, the faster the convergence will be. The homotopy-approximation of a low deformation order LHAM may not be good, nevertheless, it is better that the initial guess. If this low order homotopy-approximation is then used as an initial guess in another round of LHAM implementation, one can expect to obtain an 'even better' low order homotopy-approximation; and this process can be iterated. This is roughly the idea of iterative HAM introduced by Liao [9]. Here, we modify Liao's iterative technique such that the low order homotopy-approximation is expanded as its truncated Maclaurin series and the truncated series is then re-used as the new initial guess for the next low order approximation, and the procedure is iterated. Using this truncated reinitiated iterative approach (named RE-LHAM), a better and better LHAM approximation can be obtained in general.

Applications Examples

In this section, two test examples are solved by both LHAM and RE-LHAM to compare and to show the feasibility and advantage of RE-LAHM. We only focus on nonlinear volterra IE of the second kind having the general form

$$y(t) = f(t) + \lambda \int_a^t K(t,u) F\big(y(u)\big)\, du, \tag{11}$$

where the function $f(t)$, the kernel of the IE $K(t,u)$ and the parameter λ are given in advance while $F(y(u))$ is a nonlinear function in $y(u)$.

Test Example 1

We consider

$$y(t) = \cos t - \frac{1}{2}t - \frac{1}{4}\sin 2t + \int_0^t y^2(u)\, du \qquad (12)$$

whose solution is $\cos t$ [3]. In the frame of LHAM, the LHAM nonlinear operator for this problem is

$$\widetilde{N}\{\varphi(t,q)\} = Lap\{\varphi(t,q)\} - Lap\left\{\cos t - \frac{1}{2}t - \frac{1}{4}\sin 2t\right\} - Lap\left\{\int_0^t \varphi^2(u,q)\, du\right\}, \qquad (13)$$

Using the initial guess $v_0 = 1$ and the recurrence (10), we obtain the LHAM series approximation of this problem. For example, the series approximation under one deformation order is

$$SS_1(h,t) = 1 + \frac{1}{4}h(4 - 2t - 4\cos t + \sin 2t). \qquad (14)$$

After the 'h-curve' is plotted and studied (not shown here), $h = -1$ is chosen for this problem throughout. For example, with this h-value, the series approximations under one and two deformation order are, respectively

$$SS_1(h=-1,t) = 1 + \frac{1}{4}(-4 + 2t + 4\cos t - \sin 2t), \qquad (15)$$

$$SS_2(h=-1,t) = 1 + \frac{1}{2}(-4 + 2\,t + 4\cos t - \sin 2t) + \frac{1}{4}(3 - 10\,t + 2\,t^2 - 4\cos t + \cos 2t + 8\sin t + \sin 2t). \qquad (16)$$

For higher deformation order, one already can foresee the rapid breeding of terms in the LHAM series approximation and the raise of burden in computing $v_j(t)$ from the recurrence (10).

Implementing the RE-LHAM, a more accurate result is obtained in less CPU time. For comparison, the series approximation of 8-order LHAM ($SS_8(-1,t)$) and the series approximation of the 1-order, 8-iteration RE-LHAM ($SS_{1,8}(-1,t)$) are shown in **Figure 1** below, together with the analytic solution. The CPU time is 485 units for LHAM and 71 units for RE-LHAM.

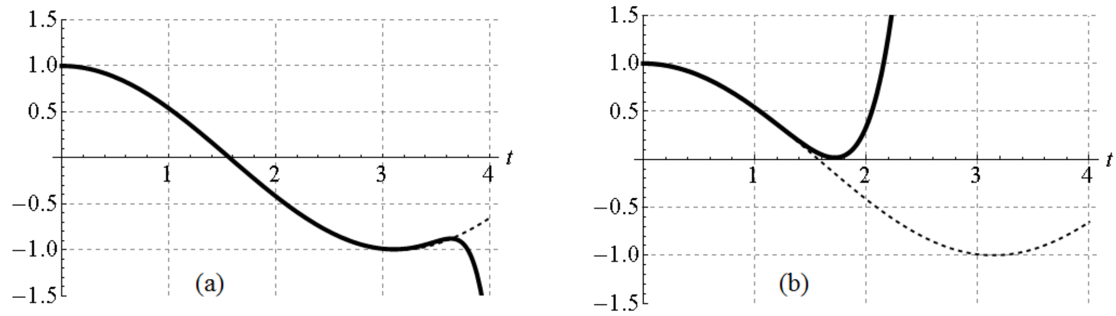

Figure 1. (a) The series approximation by RE-LHAM under 1 deformation order and 8 iterations. (b) The series approximation by the standard LHAM under 8 deformation order. The dotted curve is the exact solution $\cos t$ in both graphs.

It is obvious that the series approximation by RE-LHAM is more superior than the one by LHAM in the sense that it survives longer before diverges and furthermore, it takes much lesser CPU time. To roughly quantify how good a series approximation is, we define the '*convergence length*' as the time it takes before the absolute difference between the exact solution and the series approximation of the problem grows larger than certain tolerance. Mathematically, the convergence length, t_e, satisfies the following inequality

$$|y(t_e) - SS(t_e)| \leq \varepsilon , \tag{17}$$

where $y(t)$ is the exact solution of the problem, $SS(t)$ is the series approximation of the problem and ε is the tolerance. The convergence length simply serves as a rough measurement of how long the series approximation is 'well-behave' before diverges. For $\varepsilon = 0.5$, the convergence length by both LHAM and RE-LHAM are plotted in **Figure 2**. Apparently, the convergence length by RE-LHAM shows quite a linear increment with the number of iterations while the one by LHAM only fluctuates about 2 when the deformation order increases.

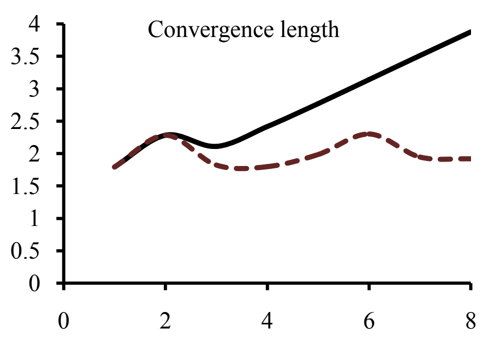

Figure 2. The convergence length versus the iteration number for RE-LHAM (solid); the convergence length versus the deformation order for LHAM (dotted).

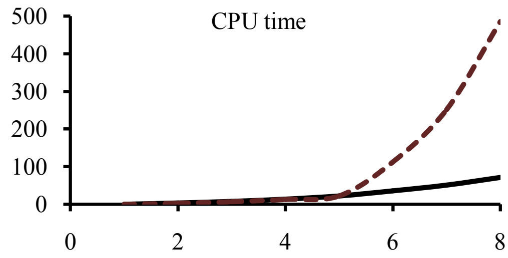

Figure 3. The CPU time versus the iteration number for RE-LHAM (solid); the CPU time versus the deformationorder for LHAM (dotted).

On the other hand, the CPU time of LHAM grows exponentially with the deformation order whereas the growth of CPU time for RE-LHAM is much slower when the iteration number increases as indicated in **Figure 3**. This example exposes the weakness of the standard LHAM, i.e. when we increase the deformation order, the computation time rises exponentially but the series approximation barely improves. This shows a very great computational inefficiency for LHAM. But then RE-LHAM has effectively served as an alternative to improve the efficiency of LHAM. It has somewhat circumvented the need to use higher deformation order to acquire better accuracy and hence avoided the complexity and time consuming issue faced in high order LHAM implementation.

Test Example 2

We consider another nonlinear Voterra IE [3]

$$y(t) = e^t + \frac{1}{2}t(e^{2t} - 1) - \int_0^t ty^2(u)\, du \tag{18}$$

having the solution $y(t) = e^t$. After taking the Laplace transform, the LHAM nonlinear operator can be readily constructed as follows

$$\tilde{N}\{\varphi(t,q)\} = Lap\{\varphi(t,q)\} - Lap\left\{e^t + \frac{1}{2}t(e^{2t} - 1)\right\} + Lap\left\{\int_0^t ty^2(u)\, du\right\}. \tag{19}$$

The initial guess $v_0(t) = 1$ is used and the series approximation can be computed using the recurrence (10). The analysis is attempted in the same direction like those in the Test Example 1: comparison is done between the 8-order LHAM series approximation and the 1-order, 8-iteration LHAM series approximation. The graphs of the series approximations and the exact solution are plotted in **Figure 4**; the convergence lengths (with $\varepsilon = 0.5$) and the CPU times are shown in **Figure 5** and **Figure 6**, respectively. In this example, the convergence-control parameter $h = -1$ is used based on the 'h-curve' study (not shown here).

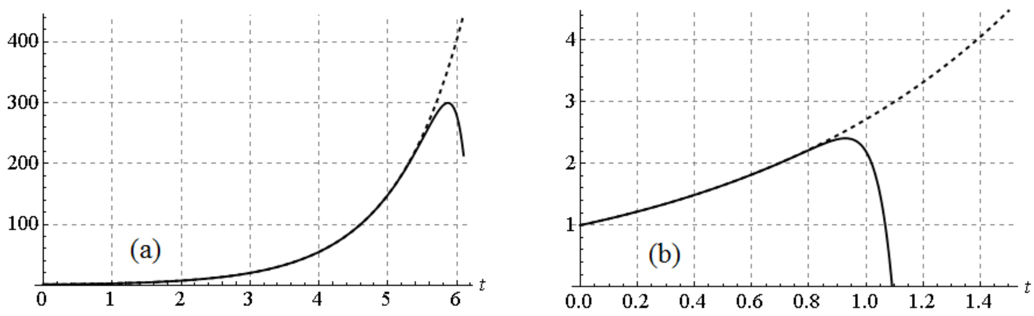

Figure 4. (a) The series approximation by RE-LHAM under 1 deformation order and 8 iterations. (b) The series approximation by the standard LHAM under 8 deformation order. The dotted curve is the exact solution e^t in both graphs

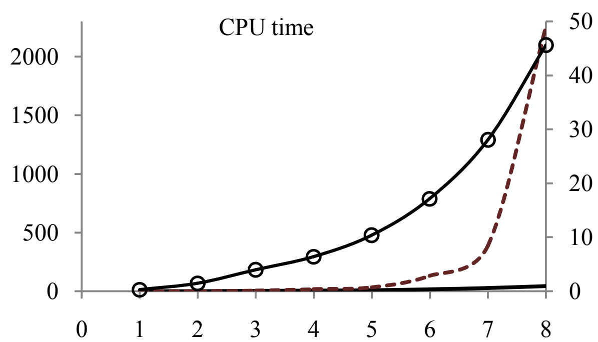

Figure 5. The CPU time versus the iteration number for RE-LHAM (solid); the convergence length versus the deformation order for LHAM (dotted). For visualization, the same convergence length for RE-LHAM is plotted again using a secondary y-axis on the right (solid with circular marker).

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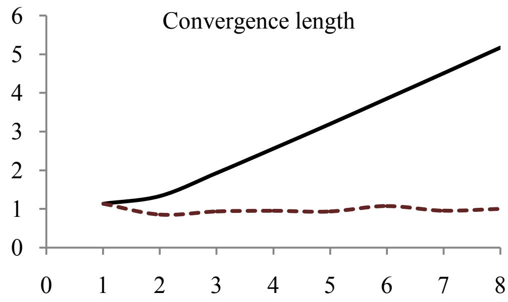

Figure 6. The convergence length versus the iteration number for RE-LHAM (solid); the convergence length time versus the deformationorder for LHAM (dotted).

For this example, the results shown in the **Figure 4-Figure 6** demonstrate a similar overview like the previous example. When the deformation order of LHAM increases, the CPU time increases exponentially but the convergence length hardly increases. The deficiency of LHAM in higher order implementation and meanwhile the validity of RE-LHAM is reaffirmed here.

Conclusions

Based on the results in the application examples, we may conclude that using LHAM, it is costly to improve the homotopy-approximation accuracy by increasing the deformation order because the CPU time rises exponentially whereas the convergence length does not increase obviously with the increase of the deformation order. However, in RE-LHAM, improving the homotopy-approximation accuracy by increasing the iteration number is more worthwhile because the CPU time rise relatively slower and yet the convergence length increase relatively much more obviously with the increase of the iteration number. Hence, the proposed RE-LHAM shows a constructive attraction in that it manages to accelerate the convergence of the homotopy-approximation meanwhile reduce the computation time and complexity of LHAM implementation as illustrated.

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