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Method for a Solution to Some Class of Quasi-Static Problems in Linear Viscoelasticity Theory, as Applied to Problems of Linear Torsion of a Prismatic Solid

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Abstract

Two theorems that reduce solutions of the general quasi-static problem of linear viscoelasticity theory to a solution of the corresponding problem of elasticity theory are proved. These theorems hold if one of the following conditions is satisfied: 1) the material is close to a mechanically uncompressible material; 2) the mean stress is zero; 3) the shift and volume hereditary functions are equal. The theorems provide free direct and inverse transforms between solutions of viscoelasticity and elasticity problems, which makes them convenient in applications. They have been applied to solutions of problems on the pure torsion of a prismatic viscoelastic solid with an arbitrary simply connected cross section. Some examples describing the obtained results have been considered.

Keywords: viscoelasticity, quasi-static problems, exact solutions, torsion problems.

1. Introduction

It is well known that solutions to the quasi-static problems in linear viscoelasticity theory, in most cases, are obtained from the corresponding problems of elasticity theory by using Volterra's principle, that is by the way of replacing elastic constants by some operators and subsequent interpretation of these operators [1]. There are also a number of methods for solving the mentioned problems by applying Laplace's, Laplace-Carson's, Fourier's integral transformations [2-4]. In the last case, an exact determination of original functions from the obtained image function is not always possible. In the present paper, under some conditions, theorems reducing solutions of quasi-static problems of linear viscoelasticity theory to solutions of the corresponding problems of elasticity theory are proved.

Statement of the General Quasi-Static Problem of Linear Viscoelasticity

We will consider an isotropic homogeneous material. Write out the determining relations between components of stress tensors σ_{ij} and deformation tensors ε_{ij} in the following form [4]:

$$2G_0 e_{ij} = s_{ij} + \int_0^t \Gamma(t-\tau) s_{ij}(\tau) d\tau, \quad (1)$$

$$K_0 \theta = \sigma + \int_0^t U(t-\tau) \sigma(\tau) d\tau, \quad (2)$$

or

$$\frac{s_{ij}}{2G_0} = e_{ij} - \int_0^t L(t-\tau) e_{ij}(\tau) d\tau, \quad (3)$$

$$\frac{\sigma}{K_0} = \theta - \int_0^t M(t-\tau) \theta(\tau) d\tau. \quad (4)$$

Here t is time; $i, j = 1, 2, 3$. Besides, $e_{ij} = \varepsilon_{ij} - \varepsilon \delta_{ij}$ is a deviation of the deformations ε_{ij}; $\varepsilon = \varepsilon_{ij} \delta_{ij} / 3$ is a mean deformation; δ_{ij} is the Kronecker symbol; $s_{ij} = \sigma_{ij} - \sigma \delta_{ij}$ is a deviation of the stresses σ_{ij}; $\sigma = \sigma_{ij} \delta_{ij} / 3$ is a mean stress; $\theta = 3\varepsilon$ is a relative variation of the volume; $G_0 = const$ is an instant elastic shift module; $K_0 = const$ is an instant elastic module of the volume deformation; the functions $\Gamma(t)$, $U(t)$, $L(t)$ and $M(t)$ are kernels of the shift creep, volume creep, shift relaxation and volume relaxation respectively.

Relations (1)-(4) are the Volterra second type integral equations. Equations (3) and (4) are derived from (1) and (2) by solving them with respect to S_{ij} and θ. In turn, equations (l) and (2) result from (3) and (4) by solving the last two ones with respect to ε_{ij} and θ. In this case, the functions $L(t)$ is a resolvent of the kernel $\Gamma(t)$ and $M(t)$ is a resolvent of the kernel $U(t)$. At the same time, the function $\Gamma(t)$ and $U(t)$ are resolvents of the kernels $L(t)$ and $M(t)$, respectively. It is clear that relations (1), (2) and (3), (4) are equivalent. Note that between the kernels $U(t)$ and $M(t)$ there exist the following integral relations [4]:

$$\Gamma(t)=L(t)+\int_0^t L(t-\tau)\Gamma(\tau)d\tau, \quad (5)$$

$$U(t)=M(t)+\int_0^t M(t-\tau)U(\tau)d\tau. \quad (6)$$

The components of the stress σ_{ij} satisfy the balance equation

$$\sigma_{ij,j}+F_i=0, \quad (7)$$

where F_i are volume forces.

Suppose that surface forces R_i are given on a part S_σ, of the boundary surface, boundary displacements u_{0i} are given on the remaining part S_u:

$$\sigma_{ij}l_j\big|_{S_\sigma}=R_i;u_i\big|_{S_u}=u_{0i}, \quad (8)$$

where l_j are direction cosines.

To solve the problem in the displacements u_i, the Cauchy geometric relations

$$\varepsilon_{ij}=\left(u_{i,j}+u_{j,i}\right)/2, \quad (9)$$

should be adjoined to the relations (1),(2) (or (3), (4)), (7), (8). Further to solve the problem in the stresses σ_{ij}, instead of (9) it is necessary to use six independent equations of deformations compatibility. By [3], one of the forms has the form

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{ik,jl}-\varepsilon_{jl,ik}=0, \tag{10}$$

where $i,j,k,l=1,2,3$.

Solution of the General Quasi-Static Problem of Linear Viscoelasticity Under the Conditions Determined

First, formulate the following lemma [5] without proof.

Lemma. For the satisfaction of the homogenous equation

$$\mu_{ij}(t)+\int_0^t r(t-\tau)\mu_{ij}(\tau)d\tau=0, \tag{11}$$

where $r(t)$ has some resolvent associated with $r(t)$ by relations of the type (5), (6), it is necessary and sufficient that

$$\mu_{ij}(t)=0,(i,j=1,2,3). \tag{12}$$

Theorem 1. Let one of the following there conditions hold 1) $K_0\to\infty$ (material is mechanically uncompressible), 2) $\sigma=0$ (the mean stress is zero, but $K<\infty$) 3) $\Gamma(t)=U(t)$ (the kernels of the shift and volume creeps coincide) or equivalently, $L(t)=M(t)$ (the kernels of the shift and volume relaxations coincide). Then the exact solution of problems (1), (2) (or (3), (4)), (7)-(10) is represented as

$$u_i=u_i^{'}+\int_0^t\Gamma(t-\tau)u_i^{'}d\tau,\sigma_{ij}=\sigma_{ij}^{'} \tag{13}$$

where $u_i^{'}$ and $\sigma_{ij}^{'}$, arc solutions of the following quasi-static problem of elasticity theory:

$$2G_0e_{ij}^{'}=s_{ij}^{'}, \tag{14}$$

$$\theta^{'}=0,\text{under conditions 1 and 2,} \tag{15}$$

or

$$K_0\theta' = \sigma', \text{under condition 3.} \tag{16}$$

$$\sigma'_{ij} + F_i = 0, \sigma'_{ij} l_j \big|_{S_\sigma} = R_i; u_i \big|_{S_u} = u'_{0i}, \tag{17}$$

$$\varepsilon'_{ij} = \left(u'_{ij} + u'_{ji}\right)/2, \varepsilon'_{ij,kl} + \varepsilon'_{kl,ij} - \varepsilon'_{ik,jl} - \varepsilon'_{jl,ik} = 0. \tag{18}$$

The following notation has been accepted:

$$e'_{ij} = \varepsilon'_{ij} - \varepsilon'\delta_{ij}, s'_{ij} = \sigma'_{ij} - \sigma'\delta_{ij}, \varepsilon' = \varepsilon'_{ij}\delta_{ij}/3, \sigma' = \sigma'_{ij}\delta_{ij}/3, \theta' = 3\varepsilon', \tag{19}$$

$$u'_{oi} = u_{oi} - \int_0^t L(t-\tau) u_{oi} d\tau. \tag{20}$$

The proof of theorem 1 is performed by direct substitution of formulae (13) into relations involved in the statement of the initial problem. In doing so, one should use lemmas (11), (12) and also the accepted notation (19), (20). If in the statement of the viscoelasticity problem, (3) and (4) are used as determining equations, then it is necessary to take advantage of relations (5), (6). Note that the deformation components ε_{ij} are defined by formula

$$\varepsilon_{ij} = \varepsilon'_{ij} + \int_0^t \Gamma(t-\tau)\varepsilon'_{ij}(\tau) d\tau, \tag{21}$$

where ε'_{ij} is expressed by u'_i in the first formula of (18).

Theorem 2. Let one of conditions 1, 2, 3 of theorem 1 hold. Then the exact solution of problem (1), (2) (or (3), (4)), (7)-(10) is represented alternatively to (13) by the formula

$$u_i = u''_i, \sigma_{ij} = \sigma''_{ij} - \int_0^t L(t-\tau)\sigma''_{ij}(\tau) d\tau, \tag{22}$$

where u''_i, σ''_{ij} are solutions of the following quasi-static problem of linear elasticity theory:

$$2G_0 e_{ij}^{''} = s_{ij}^{''}, \tag{23}$$

$$\theta^{''} = 0, \text{under conditions 1 and 2,} \tag{24}$$

or

$$K_0 \theta^{''} = \sigma^{''}, \text{under condition 3.} \tag{25}$$

$$\sigma_{ij_1 j}^{''} + F_i^{''} = 0, \sigma_{ij}^{''} l_j \Big|_{S_\sigma} = R_i^{''}; u_i^{''} \Big|_{S_u} = u_{oi}, \tag{26}$$

$$\varepsilon_{ij}^{''} = \left(u_{i,j}^{''} + u_{j,i}^{''} \right) / 2, \varepsilon_{ij,ji}^{''} + \varepsilon_{kl,lk}^{''} - \varepsilon_{ik,jl}^{''} - \varepsilon_{jl,ik}^{''} = 0. \tag{27}$$

The following notation has been accepted:

$$e_{ij}^{''} = \varepsilon_{ij}^{''} - \varepsilon^{''} \delta_{ij}, s_{ij}^{''} = \sigma_{ij}^{''} - \sigma^{''} \delta_{ij}, \varepsilon^{''} = \varepsilon_{ij}^{''} \delta_{ij} / 3, \sigma^{''} = \sigma_{ij}^{''} \delta_{ij} / 3, \theta^{''} = 3\varepsilon^{''}, \tag{28}$$

$$F_i^{''} = F_i + \int_0^t \Gamma(t-\tau) F_i d\tau; R_i^{''} = R_i + \int_0^t \Gamma(t-\tau) R_i d\tau. \tag{29}$$

The proof of theorem 2 is performed by the direct substitution of formula (22) into all the necessary relations. In this case, notation (28), (29), lemma (11), (12) and also relations (5), (6) between the resolvent functions.

Let now a solution of the elasticity problem is known with respect to the volume force F_i^e, the surface force R_i^e and the boundary displacement u_i^e: $u_i^e = u_i^e\left(F_i^e, R_i^e, u_{oi}^e\right)$, $\varepsilon_{ij}^e = \varepsilon_{ij}^e\left(F_i^e, R_i^e, u_{oi}^e\right)$, $\sigma_{ij}^e = \sigma_{ij}^e\left(F_i^e, R_i^e, u_{oi}^e\right)$. By using theorem 1 and changing F_i^e, R_i^e and u_{oi}^e to F_i, R_i and $u_{oi}^{'}$, respectively, we determine $u_i^{'}$, $\varepsilon_{ij}^{'}$, $\sigma_{ij}^{'}: u_i^{'} = u_i^e\left(F_i, R_i, u_{oi}^{'}\right)$, $\varepsilon_{ij}^{'} = \varepsilon_{ij}^e\left(F_i, R_i, u_{oi}^{'}\right)$, $\sigma_{ij}^{'} = \sigma_{ij}^e\left(F_i, R_i, u_{oi}^{'}\right)$, where $u_{oi}^{'}$ is defined in (20).

It is also necessary to make change of the elasticity module G and K, to G_0 and K_0, respectively. After determining the quantities $u_i^{'}$, $\varepsilon_{ij}^{'}$, $\sigma_{ij}^{'}$, by formula (13), we find the sought - for u_i,

σ_{ij}. Sought - for components of the deformation tensor ε_{ij} are determined by formula (21). If we make use of the theorem 2, the quantities u_o'', ε_{ij}'', σ_{ij}'' will be: $u_i'' = u_i^e\left(F_i'', R_i'', u_{oi}\right)$, $\varepsilon_{ij}'' = \varepsilon_{ij}^e\left(F_i'', R_i'', u_{oi}\right)$, $\sigma_i'' = \sigma_i^e\left(F_i'', R_i'', u_{oi}\right)$, where F_i'', R_i'' are defined by formula (29). In this case, the change of the corresponding material constants is also necessary. After finding u_i'', ε_{ij}'', σ_{ij}'', we determine the sought - for quantities u_i, σ_{ij} from formula (22). The sought – for components of the deformation will be: $\varepsilon_{ij} = \varepsilon_{ij}''$.

2. Application

Solving the Problem of Linear Torsion of a Prismatic Viscoelastic Solid with an Arbitrary Cross Section

The problem of linear torsion of a prismatic viscoelastic solid serves a good example for applications of the above - formulated theorems. Because in this case, one of the conditions of these theorems, namely the condition that the mean stress is zero, is fulfilled.

Let the forces leading to braiding couples, re applied to the base of a prismatic viscoelastic solid with an arbitrary cross section. We will suppose that the side surface of the solid is free of external forces and volume forces me abscent. Mechanical properties of the material of a prismatic solid are characterized by relations (1), (2) or (3), (4) of linear viscoelasticity theory.

We use the Cartesian coordinate system $\left(x_1, x_2, x_3\right)$. Direct the axis x_3 parallel to the axis of the prismatic solid. At not constrained (pure) torsion of a prismatic viscoelastic solid with an arbitrary cross section, according to Saint-Venant we consider that in a fixed period of time 1) equally distant cross sections twist at the same angles; 2) all the cross sections are equally bent; deplanatiens $\left(u_3\right)$ proportionally depending n time of torsional angle are emerged, which is allowable in linear torsion. Write out mathematically the mentioned assumptions in the form:

$$u_1 = -\gamma\left(t\right) x_2 x_3; u_2 = \gamma\left(t\right) x_1 x_3; u_3 = \gamma\left(t\right)\varphi\left(x_1, x_2\right). \quad (30)$$

Here $\varphi(x_1, x_2)$ is a function of deplanation, $\gamma = \gamma(t)$ is a relative angle of torsion at the instant of time t. In case of $\gamma(t) \equiv const$, relations (30) coincide with the corresponding relations of Saint-Venant [6].

Using properties of heredity, which are applicable to viscoelastic solids, we represent the function $\gamma(t)$ in the form

$$\gamma(t) = \vartheta(t) + \int_0^t \Gamma(t-\tau)\vartheta(t)\,d\tau, \tag{31}$$

where $\vartheta(t)$ is some of time function to be determined.

Considering (30) in (9), we have

$$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = \varepsilon_{12} = 0; \tag{32}$$

$$\varepsilon_{13} = \frac{\gamma(t)}{2}\left(\frac{\partial\varphi}{\partial x_1} - x_2\right); \varepsilon_{23} = \frac{\gamma(t)}{2}\left(\frac{\partial\varphi}{\partial x_2} + x_1\right). \tag{33}$$

Using relations (32), (33) in equations (3), (4), determine the quantities σ_{ij}:

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma = 0; \tag{34}$$

$$\sigma_{13} = G_0\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\left[\gamma(t) - \int_0^t R(t-\tau)\gamma(t)\,d\tau\right]; \tag{35}$$

$$\sigma_{23} = G_0\left(\frac{\partial\varphi}{\partial x_2} + x_1\right)\left[\gamma(t) - \int_0^t R(t-\tau)\gamma(t)\,d\tau\right]. \tag{36}$$

Taking into account (31) and (5), relations (35) and (36) turn into the form

$$\sigma_{13} = G_0\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\vartheta(t), \sigma_{23} = G_0\left(\frac{\partial\varphi}{\partial x_2} + x_2\right)\vartheta(t). \tag{37}$$

From the balance equations only the followings are omitted:

$$\frac{\partial \sigma_{13}}{\partial x_3}=0,\frac{\partial \sigma_{23}}{\partial x_3}=0,\frac{\partial \sigma_{13}}{\partial x_1}+\frac{\partial \sigma_{23}}{\partial x_2}=0. \tag{38}$$

The first two equations of (38) are satisfied as an identity the thir4 if (37) is taken into account, yields

$$\frac{\partial^2 \varphi}{\partial x_1^2}+\frac{\partial^2 \varphi}{\partial x_2^2}\equiv \nabla \varphi=0. \tag{39}$$

Formula (39) shows that on the domain occupied by the cross section of solid the deplanetion function $\varphi(x_1,x_2)$ must be a harmonic function of the variables x_1 and x_2. It follows from the last argument that the deplanation itself should also be a harmonic function.

In the considered case, as in theory of elastic torsion, it can be shown that on contour Ω of the cross section, the deplanation function p satisfies the condition

$$\frac{\partial \varphi}{\partial n}=\left[x_2\cos(n,x_1)-x_1\cos(n,x_2)\right]_\Omega$$

or

$$\frac{\partial \varphi}{\partial n}=\frac{d}{ds}\left(\frac{x_1^2+x_2^2}{2}\right)\Bigg|_\Omega, \tag{40}$$

where $\frac{d}{dn}$, $\frac{d}{ds}$ are derivatives with respect to the normal n and the are Ω respectively.

Thus, the problem of viscoelastic prismatic solid's torsion, in similar way as the problem of elastic solid's torsion, is reduced to the Neuman problem (39), (40) for Laplaee equation. In this case, it can be shown that existence conditions for a solution of the Neuman problem $\int\limits_\Omega \frac{\partial \varphi}{\partial n}ds=0$ are fulfilled.

For stresses equally acting on the face surface we have

$$\omega\int \sigma_{13}d\omega=0,\omega\int \sigma_{23}d\omega=0, \tag{41}$$

where ω is an area of the cross section of a prismatic solid.

Taking (41) into account, we come to a conclusion that tangent stresses applied to the cross section are reduced to a pair of force which has the moment

$$M(t) = \omega \int (x_1\sigma_{23} - x_2\sigma_{13})\, d\omega. \tag{42}$$

The balance condition on the face surface gives $M(t) = M_T(t)$, where $M_T(t)$ is the given twisting moment. Considering this and formulas (37) in relation (42), we obtain that

$$\vartheta(t) = \frac{M_T(t)}{D} \tag{43}$$

where $D = G_0\omega\int\left(x_1^2 + x_2^2 + x_1\frac{\partial\varphi}{\partial x_2} - x_2\frac{\partial\varphi}{\partial x_1}\right)d\omega$ is a rigidity in torsion. It can be show that always $D > 0$.

Therefore, the problem of physical linear torsion of a viscoelastic prismatic solid is completely solved if we find the deplanation function $\varphi(x_1, x_2)$.

Now represent the solution of linear torsion problem in the form (13). In this case the quantities ε_{ij} and $\varepsilon_{ij}^{'}$ are expressed by formulas (21) and the first formula of (18), respectively.

The balance equations (38), in view of the second formula of (13), maintain their form:

$$\frac{\partial\sigma_{13}^{'}}{\partial x_3} = 0, \frac{\partial\sigma_{23}^{'}}{\partial x_3} = 0, \frac{\partial\sigma_{13}^{'}}{\partial x_1} + \frac{\partial\sigma_{23}^{'}}{\partial x_2} = 0. \tag{44}$$

Besides, from (34) and (37) we obtain:

$$\sigma_{11}^{'} = \sigma_{22}^{'} = \sigma_{33}^{'} = \sigma_{12}^{'} = \sigma^{'} = 0$$

$$\sigma_{13}^{'} = G_0\left(\frac{\partial\varphi}{\partial x_1} - x_2\right)\vartheta(t), \sigma_{23}^{'} = G_0\left(\frac{\partial\varphi}{\partial x_2} + x_1\right)\vartheta(t). \tag{45}$$

Using (21) and (31), formulas (32) and (33) are converted into the form

$$\varepsilon_{11}^{'} = \varepsilon_{22}^{'} = \varepsilon_{33}^{'} = \varepsilon_{12}^{'} = 0$$

$$\varepsilon_{13}^{'} = \frac{\vartheta(t)}{2}\left(\frac{\partial\varphi}{\partial x_1} - x_2\right), \varepsilon_{23}^{'} = \frac{\vartheta(t)}{2}\left(\frac{\partial\varphi}{\partial x_2} + x_1\right). \tag{46}$$

From formula (30) for components of the displacement vector, in view of the first formula of (13) and also formula (31), it follows that

$$u_1' = -\vartheta(t) x_2 x_3; u_2' = \vartheta(t) x_1 x_3; u_3' = \vartheta(t) \varphi(x_1 x_2). \tag{47}$$

Relation (42) with consideration of formula (13) will be written in the form

$$M_T(t) = M(t) = \omega \int \left(x_1 \sigma_{23}' - x_2 \sigma_{13}' \right) d\omega \tag{48}$$

Relation (43) does not change its form.

As we see, relations (43)-(48) are relations of elastic quasi-static torsion theory. This means that as applied to the problem to torsion, the quantities u_i', σ_{ij}' and ε_{ij}' involved in formulas (13), (21) are components of displacement vector, stress tensors and deformations, respectively, which appear in the considered prismatic solid while its quasi-static elastic torsion with the torsion moment $M_T(t)$. In this case, t plays the role of only a parameter.

Therefore, if by any one of the existing methods, the problem of elastic torsion of a prismatic solid with the given cross section has been solved under the condition that the displacement module G and the torsion moment M are known, that is the elastic displacements u_i^e, deformations ε_{ij}^e, stresses σ_{ij}^e have been found then by changing G to G_0, M to $M_T(t)$ in the expressions of u_i^e, ε_{ij}^e, σ_{ij}^e, we find the quantities u_i', ε_{ij}', σ_{ij}'. Thereafter according to formulas (13) and (21), we determine the sought for solution of the corresponding problem of viscoelasticity.

At this point remark that by [6], while solving the considered problems of elastic and viscoelastic torsion, instead of the deplanation function φ one can use either the harmonic function ψ, conjugate to φ, or the Prandtle torsion function Φ which is associated with ψ by the relation $\Phi(x_1, x_2) = \psi(x_1, x_2) - \left(x_1^2 + x_2^2\right)/2$. In this case, as is known from [6], the problems of determining the functions $\psi(x_1, x_2)$ and $\Phi(x_1, x_2)$ are the Dirichlet problems for Laplace's equation. When solving the mentioned problems, the complex function of torsion [6] can also be used.

3. Examples

1. Torsion of a Prismatic Beam with an Elliptic Cross Section

Let a and b be semiaxes of an ellipse. We draw on the solution of the corresponding problem of elasticity [6]:

$$u_1^e = -\frac{M\left(a^2+b^2\right)x_2x_3}{G\pi a^3b^3}, u_2^e = \frac{M\left(a^2+b^2\right)x_1x_3}{G\pi a^3b^3}$$

$$u_3^e = \frac{M\left(b^2-a^2\right)x_1x_2}{G\pi a^3b^3}; \sigma_{13}^e = -\frac{2M}{\pi ab^3}x_2; \sigma_{23}^e = \frac{2M}{\pi a^3b}x_1,$$

where G is a module of the material shift, M is a moment of torsion.

Replacing in the last expressions G by G_0, M by $M_T\left(t\right)$, we will have expressions for the quantities $u_1^{'}$, $u_2^{'}$, $u_3^{'}$, $\sigma_{13}^{'}$, $\sigma_{23}^{'}$. Considering the obtained expression in transitional formula (13), write out the solution to the problem of torsion of an viscoelastic prismatic solid with an elliptic cross section:

$$u_1 = -\frac{\left(a^2+b^2\right)x_2x_3}{G_0\pi a^3b^3}M_T^*\left(t\right); u_2 = \frac{\left(a^2+b^2\right)x_1x_3}{G_0\pi a^3b^3}M_T^*\left(t\right);$$

$$u_3 = \frac{\left(b^2-a^2\right)x_1x_2}{G_0\pi a^3b^3}M_T^*\left(t\right); \sigma_{13} = -\frac{2M_T\left(t\right)}{\pi ab^3}x_2; \sigma_{23} = \frac{2M_T\left(t\right)}{\pi a^3b}x_1.$$

Here

$$M_T^*\left(t\right) = M_T\left(t\right) + \int_0^t \Gamma\left(t-\tau\right)M_T\left(\tau\right)d\tau. \tag{49}$$

In case of $a = b$, the obtained solution corresponds to the solution of the torsion problem related to a viscoelastic prismatic solid with a circular cross section. In this case, $u_3 = 0$, which shows an absence of deplanation.

2. Round Prismatic Beam with a Semicircular Longitudinal Bore

According to [6], the solution to the problem of elasticity is represented in the form

$$u_1^e = -\frac{Mx_2x_3}{2GDa^4}; u_2^e = \frac{Mx_1x_3}{2GDa^4}; u_3^e = -\frac{Mb^2x_2}{2GDa^3\left(x_1^2+x_2^2\right)};$$

$$\sigma_{13}^e = \frac{M}{2Da^4}\left[\frac{2ab^2x_1x_2}{\left(x_1^2+x_2^2\right)^2} - x_2\right]; \sigma_{23}^e = \frac{M}{2Da^4}\left[-\frac{ab^2\left(x_1-x_2\right)}{\left(x_1^2+x_2^2\right)^2} + x_1 - a\right].$$

Here, as in the previous problem, G is a module of the material shift, M is a moment of torsion, a is a radius of the beam disk, b is a radius of the bore disk. Besides,

$$D = \frac{1}{24}\left(\sin 4\alpha + 8\sin 2\alpha + 12a\right) - \frac{1}{2}\left(\frac{b}{a}\right)^2\left(\sin 2\alpha + 2\alpha\right) + \frac{4}{3}\left(\frac{b}{a}\right)^3 \sin\alpha,$$

where $\alpha = \arccos\left(\frac{b}{2a}\right)$.

Now to obtain a solution to the problem of torsion of a circular viscoelastic beam with a semicircular longitudinal bore, replace G by G_0, M by $M_T(t)$ in the represented solution of the problem of elasticity and use formula (13).

Then we obtain:

$$u_1 = -\frac{x_2x_3}{2G_0Da^4}M_T^*; u_2 = \frac{x_1x_3}{2G_0Da^4}M_T^*; u_3 = -\frac{b^2x_2}{2G_0Da^3\left(x_1^2+x_2^2\right)}M_T^*$$

$$\sigma_{13} = \frac{M_T(t)}{2Da^4}\left[\frac{2ab^2x_1x_2}{\left(x_1^2+x_2^2\right)^2} - x_2\right]; \sigma_{23} = \frac{M_T(t)}{2Da^4}\left[-\frac{ab^2\left(x_1^2-x_2^2\right)}{\left(x_1^2+x_2^2\right)^2} + x_1 - a\right].$$

The operator u_3 involved in these relations has the form (49).

4. Remarks

The problem of torsion of a prismatic viscoelastic solid with an arbitrary simply connected cross section has been solved. Problems of viscoelasticity for prismatic solids with multiply connected cross sections can be solved in the similar way.

5. Conclusion

l. Formulas reducing solutions of the general quasi-static problem of linear viscoelasticity for an isotropic and homogeneous solid to a solution of the corresponding problem of elasticity theory are presented. These formulas are valid if one of the following conditions holds: l) the material is close to a mechanically uncompressible matenal; 2) the mean stress is zero; 3) the shift and volume hereditary functions are equal. They provide a free conversion from the problem of viscoelasticity to the problem of elasticity and vice versa. This quality makes them convenient in applications.

2. The obtained result has been applied to a solution of the problem of pure torsion for a prismatic viscoelastic solid with an arbitrary simply connected cross section. Examples describing the construction procedure for a solution of the problem of viscoelasticity from the known solutions of the corresponding problem of elasticity have been presented.

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