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On the New Type Almost Sequence Space

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Abstract

In this paper, we introduce *trf* sequence spaces by means of the matrix domain of $B(r, s, t)$ triple band matrix and f_T defined by Zararsız [9]. Furthermore, we determine β- and γ- duals of the space *trf* and characterize the classes $(trf:\ell_\infty)$, $(trf:c)$, $(\ell_\infty:trf)$ and $(c:trf)$.

Keywords: Almost convergence, β- and γ- duals, matrix domain of a sequence space, isomorphism.

1. Preliminaries, Background and the Notation

The notion of almost convergence was introduced by Lorentz [5]. It impressed mathematicians to construct several types of classes of sequence spaces. Throughout the paper, w, the space of all complex valued sequences, is called a sequence space. The notations ℓ_∞, c, c_0, ℓ_p, f and f_0 are showed for the sequence spaces of all bounded, convergent, null, absolutely p- summable, almost convergent and almost null convergent sequences, respectively. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. Let λ and μ be two sequence spaces and $A=(a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where n,k are positive integers. Then, A defines a matrix mapping from λ to μ and is denoted by $A:\lambda\to\mu$ if for every sequence $x=(x_k)\in\lambda$ the sequence $Ax=\{(Ax)_n\}$, the A transform of x, is in μ where

$$(Ax)_n=\sum_k a_{nk}x_k, n\in\mathbb{N}. \tag{1}$$

By $(\lambda:\mu)$, we denote the class of matrices A such that $A:\lambda\to\mu$. Thus, $A\in(\lambda:\mu)$ if and only if the series on the right side of (1) converges for each $n\in\mathbb{N}$ and every $x\in\lambda$, and we have $Ax=\{(Ax)_n\}_{n\in\mathbb{N}}\in\mu$ for all $x\in\lambda$. The domain λ_A of an infinite matrix A in a sequence space λ is defined by

$$\lambda_A=\{x=(x_k)\in w: Ax\in\lambda\}.$$

For brevity in notation, through all the text, we shall write $\sum_n$, $\sup_n$, $\lim_n$ and $\triangle a_{nk}$ instead of $\sum_{n=0}^{\infty}$, $\sup_{n\in\mathbb{N}}$, $\lim_{n\to\infty}$ and $a_{nk}-a_{n,k+1}$. Furthermore, we write $\mathbb{R}$ and $\mathbb{C}$ for the set of real or complex valued numbers, respectively.

The Cesàro matrix of order one which is a lower triangular matrix defined by the matrix $C=(c_{nk})$ as follows:

$$c_{nk}=\begin{cases}\dfrac{1}{n+1} & , \quad 0\le k\le n,\\ 0 & , \quad k>n,\end{cases}$$

for all $n,k\in\mathbb{N}$.

One of the best known regular matrix is $R=(r_{nk})$, the Riesz matrix which is a lower triangular matrix defined by

$$r_{nk}=\begin{cases}\dfrac{r_k}{R_n} & , \quad 0\le k\le n,\\ 0 & , \quad k>n,\end{cases}$$

for all $n,k\in\mathbb{N}$, where (r_k) is real sequence with $r_0>0$, $r_k\ge 0$ and $R_n=\sum_{k=0}^{n} r_k$. The Riesz matrix R is regular if and only if $R_n\to\infty$ as $n\to\infty$, [7].

Let r, s and t be non-zero real numbers, and define the triple matrix $B(r,s,t) = \{b_{nk}(r,s,t)\}$ for all $k, n \in \mathbb{N}$ as follows:

$$b_{nk}(r,s,t) = \begin{cases} r, & \mathrm{k = n;} \\ s, & \mathrm{k = n-1;} \\ t, & \mathrm{k = n-2;} \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to calculate that the inverse $B^{-1}(r,s,t) = \{b_{nk}^{-1}(r,s,t)\}$ of the triple band matrix is given by

$$\{b_{nk}^{-1}(r,s,t)\} = \begin{cases} \dfrac{1}{r}\displaystyle\sum_{m=0}^{n-k}\left(\dfrac{-s+\sqrt{s^2-4tr}}{2r}\right)^{n-k-m}\left(\dfrac{-s-\sqrt{s^2-4tr}}{2r}\right)^{m}, & 0 \le k \le n; \\ 0, & k > n. \end{cases}$$

for all $k, n \in \mathbb{N}$. If we consider of the value $t = 0$, then we obtain $B(r,s,t) = B(r,s)$ named difference generalized matrix. From here, the consequences concern to matrix domain of the $B(r,s,t)$ are more extensive than the results of Zararsız [8].

The rest of the present paper is organized, as in the following:

2. Almost Convergent Sequences Space *f*

In this section, we deal with almost convergent sequences space f. We begin with writing some required definitions and lemma by means of Lorentz [5].

The shift operator S on ℓ_∞ is defined by $(Sx)_n = x_{n+1}$ for all $n \in \mathbb{N}$. A Banach limit L is a non-negative linear functional on ℓ_∞ satisfying $L(Sx) = L(x)$ and $L(e) = 1$ where $e = (1,1,1,...)$. Any bounded sequence is called almost convergent to the generalized limit a if all Banach limits of the sequence x are equal to a [5]. This is denoted by $f - \lim x = a$. It is given by Lorentz [5] that

$f-\lim x=a$ if and only if $\lim_p \frac{(x_n+x_{n+1}+...+x_{n+p-1})}{p}=a$, uniformly in n. By f and f_0, we denote the space of all almost convergent and almost null sequences, respectively, i.e.,

$$f=\left\{x=(x_k)\in \ell_\infty : \exists a\in \mathbb{C} \ni \lim_m \sum_{k=0}^{m} \frac{x_{n+k}}{m+1}=a,\ uni.in\ \ n\right\}$$

and

$$f_0=\left\{x=(x_k)\in \ell_\infty : \lim_m \sum_{k=0}^{m} \frac{x_{n+k}}{m+1}=0,\ uni.in\ \ n\right\}.$$

Lorentz [6] obtained the necessary and sufficient conditions for an infinite matrix to contain f in its convergence domain. These conditions are standard Silverman Toeplitz conditions for regularity and plus the following condition

$$\lim_n \sum_{k=0}^{\infty} | a_{nk}-a_{n,k+1} |=0. \tag{2}$$

A matrix U is called the *generalized Cesàro matrix* if it is obtained from the matrix C by shifting rows. Let $\theta:\mathbb{N}\to\mathbb{N}$. Then $U=(u_{nk})$ is defined by

$$u_{nk}=\begin{cases} \dfrac{1}{n+1} & , \quad \theta(n)\le k\le \theta(n)+n, \\ 0 & , \quad \text{otherwise,} \end{cases}$$

for all $n,k\in\mathbb{N}$.

Let us suppose that G is the set of all such matrices obtained by using all possible functions θ. Now, right here, let's give a new lemma for the set of almost convergent sequences which was given by Butkovic, Kraljevic and Sarapa, [4]:

Lemma 1. The set f of all almost convergent sequences is equal to the set $\cap_{U\in G} c_U$.

3. The Sequence Spaces *trf* Derived by the Domain of the Triple Band Matrix

In this section, we wish to introduce the new spaces trf and trf_0 as the sets of all sequences such that their $B(r,s,t)$ transforms are in the spaces rf and rf_0, respectively. Later, we give an isomorphism between the spaces trf, trf_0 and rf, rf_0, respectively. Furthermore, we define a norm on the spaces trf and trf_0 and show that these spaces are Banach with this norm. Finally, we investigate some algebraic properties on the spaces rf and rf_0 and trf and trf_0.

The definition of almost convergence can be defined as the intersection of convergence field that is obtained by displacement of the lines of first-order Cesàro matrix. Let $v \in \mathbb{N}$ and $x = (x_k) \in \ell_\infty$. Let us define the matrix $S^v = (s_{nk}^v)$ as follows:

$$s_{nk}^v = \begin{cases} 1 & , \quad n + v = k, \\ 0 & , \quad others. \end{cases}$$

The sequence $(S^v x) = (S^0 x, S^1 x, S^2 x, ..., S^v x, ...)$ named shifted transforms sequence of x, is obtained by S. Thus, almost convergence has the same meaning with the convergence of first-order Cesàro average of the shifted transform sequence $(S^v x) = (S^0 x, S^1 x, S^2 x, ..., S^v x, ...)$ to a fixed sequence for each v. After these, we can generalize the set of almost convergent and almost null sequence spaces by the following sequence spaces called as the set of all T - convergent and null T - convergent sequences, respectively:

$$f_T = \left\{ x \in \ell_\infty : \lim_k [T(S^v x)]_k = \ell \in \mathbb{C}, v = 0,1,2,... \right\}$$

$$f_{T_0} = \left\{ x \in \ell_\infty : \lim_k [T(S^v x)]_k = 0, v = 0,1,2,... \right\}.$$

By taking $R = (r_{nk})$ instead of matrix T on the sets f_T and f_{T_0}, respectively, rf - convergent and null rf - convergent sequences spaces are defined by Zararsız [8] as follows, i.e.,

$$rf = \left\{ x = (x_k) \in \ell_\infty : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} = a,\ uni.\ in\ n \right\} \tag{3}$$

$$rf_0 = \left\{ x = (x_k) \in \ell_\infty : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} = 0,\ uniformly\ in\ n \right\} \tag{4}$$

Now, we define two original spaces of convergent sequences, showed as trf and trf_0 as the sets of all sequences such that their $B(r,s,t)$- transforms are in the spaces rf and rf_0, respectively, it means that;

$$trf = \left\{ x = (x_k) \in w : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} U(n,k) = a,\ uni.\ in\ n \right\} \tag{5}$$

$$trf_0 = \left\{ x = (x_k) \in w : \lim_m \frac{1}{R_m} \sum_{k=0}^{m} U(n,k) = 0,\ uni.\ in\ n \right\}, \tag{6}$$

where $U(n,k) = r_k[tx_{k+n-2} + sx_{k+n-1} + rx_{k+n}]$.

Let us define the sequence $y = (y_k)$, as the $B(r,s,t)$- transform of a sequence $x = (x_k)$ as follows:

$$y_k = tx_{k-2} + sx_{k-1} + rx_k, (k \in \mathbb{N}). \tag{7}$$

Now, we give a Lemma as follows which is necessary for us:

Lemma 2. [8] The sets rf and rf_0 are Banach spaces with the norm

$$\| x \|_{rf} = \| x \|_{rf_0} = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k x_{k+n} \right|, uniformly\ in\ n. \tag{8}$$

Corollary 1. [8] The space rf has no Schauder basis.

Theorem 1. Define the norm on the sets trf and trf_0 as follows:

$$\| x \| = \sup_m \left| \frac{1}{R_m} \sum_{k=0}^{m} r_k [t x_{k+n-2} + s x_{k+n-1} + r x_{k+n}] \right|, \textit{uniformly in n.} \tag{9}$$

Then the sets trf and trf_0 are linear spaces with the co-ordinatewise addition and scalar multiplication.

Proof. It is clear the property of that rf and rf_0 are Banach spaces and $B(r,s,t)$ is normal matrix. □

Theorem 2. The sequence spaces rf and rf_0 are linearly isomorphic to the spaces trf and trf_0, respectively.

Proof. Consider the transformation F defined using the notation of (7), from trf to rf, by $x \to y = Fx$. The linearity of F is clear. Let $y = (y_k) \in rf$ and define the sequence $x = (x_k)$ by $(\{B^{-1}(r,s,t)y\})_k$ for all $k \in \mathbb{N}$. Then, it is clear that;

$$\{B(r,s,t)x\}_k = t x_{k-2} + s x_{k-1} + r x_k = y_k$$

for all $k \in \mathbb{N}$ which shows that

$$f - \lim \{B(r,s,t)x\}_k = \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k (t x_{k-2} + s x_{k-1} + r x_k) \tag{10}$$

$$= \lim_m \frac{1}{R_m} \sum_{k=0}^{m} r_k y_{k+n} \tag{11}$$

$$= trf - \lim y_k, \ \textit{uniformly in n.} \tag{12}$$

It means that $x = (x_k) \in trf$. Namely, F is surjective. Because of the fact that F is a linear bijection, trf and rf are linearly isomorphic. This completes the proof. □

4. Duals

In this section, we determine the β - and γ - duals of the spaces trf and trf_0. For the sequence spaces λ and μ, define the set $S(\lambda,\mu)$ by

$$S(\lambda,\mu)=\{z=(z_k)\in w: xz=(x_k z_k)\in\mu \text{ for all } x=(x_k)\in\lambda\}. \tag{13}$$

If we take $\mu=\ell_1$ then the set $S(\lambda,\ell_1)$ is called α - dual of λ and similarly the sets $S(\lambda,cs)$, $S(\lambda,bs)$ are called β - and γ - duals of λ and denoted by λ^β, λ^γ, respectively.

We can give the following lemmas and proposition which will be used in the computation of the β - dual of the sets trf and trf_0.

Lemma 3. [1] Let $A=(a_{nk})$ be an infinite matrix for all $k,n\in\mathbb{N}$. Then $A\in(rf:\ell_\infty)$ if and only if

$$\sup_n\sum_k |a_{nk}|<\infty. \tag{14}$$

Proposition 1. Let $A=(a_{nk})$ be an infinite matrix for all $k,n\in\mathbb{N}$. Then $A\in(rf:c)$ if and only if

$$\lim_n\sum_k a_{nk}=a, a\in\mathbb{R} \tag{15}$$

$$\lim_n a_{nk}=a_k,(a_k\in\mathbb{C},k\in\mathbb{N}), \tag{16}$$

$$\lim_n\sum_k |\triangle(a_{nk}-a_k)|=0 \tag{17}$$

hold.

Lemma 4. Let $A=(a_{nk})$ be an infinite matrix for all $k,n\in\mathbb{N}$. Then $A\in(\ell_\infty:rf)$ if and only if (14) and

$$rf-\lim_{n} a_{nk}=a_k, \forall k\in\mathbb{N}, \tag{18}$$

$$\lim_{m}\sum_{k}\left|\frac{1}{R_m}\sum_{i=0}^{m} r_i a_{n+i,k}-a_k\right|=0, uniformly in\ n \tag{19}$$

hold.

Lemma 5. Let $A=(a_{nk})$ be an infinite matrix for all $k,n\in\mathbb{N}$. Then $A\in(c:rf)$ if and only if

$$\sup_{m}\sum_{k}\left|\frac{1}{R_m}\sum_{i=0}^{m} r_i a_{ik}\right|<\infty,\ (k,m\in\mathbb{N}), \tag{20}$$

$$\lim_{m}\frac{1}{R_m}\sum_{i=0}^{m} r_i a_{n+i,k}=a_k,\ uniformly\ in\ n,\ (a_k\in\mathbb{C}) \tag{21}$$

and

$$\lim_{m}\frac{1}{R_m}\sum_{k}\sum_{i=0}^{m} r_i a_{n+i,k}=a,\ uniformly\ in\ n, \tag{22}$$

hold.

Lemma 6. [1] Let $D=(d_{nk})$ be defined via a sequence $a=(a_k)\in w$ and the inverse matrix $V=(v_{nk})$ of the triangle matrix $U=(u_{nk})$ by

$$d_{nk}=\begin{cases}\sum_{j=k}^{n} a_j v_{jk} & , \quad 0\le k\le n,\\ 0 & , \quad k>n,\end{cases}$$

for all $k,n\in\mathbb{N}$. Then,

$$\{\lambda_U\}^{\gamma}=\{a=(a_k)\in w: D\in(\lambda:\ell_\infty)\}$$

and

$$\{\lambda_U\}^{\beta}=\{a=(a_k)\in w: D\in(\lambda:c)\}.$$

Theorem 3. The γ - dual of the space trf is the set d_1, where

$$d_1 = \left\{ (a_k) \in w : \sup_n \sum_{k=0}^{n} \left| \sum_{j=k}^{n} b_{jk}^{-1} a_j \right| \right\}. \tag{23}$$

Proof. The proof of the theorem is clear, so we omit it. □

Theorem 4. Let us write the sets d_2, d_3, d_4 and d_5 by

$$d_2 = \left\{ (a_k) \in w : \lim_n \sum_{j=k}^{n} b_{jk}^{-1} a_j \ \ exists \right\}, \tag{24}$$

$$d_3 = \left\{ (a_k) \in w : \lim_n \sum_{k=0}^{n} \left[\sum_{j=0}^{k} b_{jk}^{-1} \right] a_k \ \ exists \right\}, \tag{25}$$

$$d_4 = \left\{ (a_k) \in w : \lim_n \sum_{k=0}^{n} \left| \sum_{j=n}^{\infty} b_{jk}^{-1} a_j \right| = 0 \right\}, \tag{26}$$

$$d_5 = \left\{ (a_k) \in w : \lim_n \sum_{k=n+1}^{\infty} \left| \sum_{j=n+1}^{\infty} (b_{jk}^{-1} - b_{j,k+1}^{-1}) a_j \right| = 0 \right\}, \tag{27}$$

for all $j,k \in \mathbb{N}$. Then, $\{trf\}^{\beta} = \bigcap_{i=1}^{5} d_i$.

Proof. Define the matrix $V = (v_{nk})$ via the sequence $u = (u_k) \in w$ by

$$v_{nk} = \begin{cases} \sum_{j=k}^{n} b_{jk}^{-1} u_j & , \quad (0 \le k \le n), \\ 0 & , \quad (k > n), \end{cases}$$

for all $n,k \in \mathbb{N}$. By considering the relation $x_k = \sum_{j=k}^{n} b_{jk}^{-1} y_j$, we realize that

$$\sum_{k=0}^{n} u_k x_k = \sum_{k=0}^{n} \sum_{j=k}^{n} b_{jk}^{-1} u_j y_k = (Vy)_n, (n \in \mathbb{N}). \tag{28}$$

From (28), we see that $ux=(u_k x_k)\in cs$ whenever $x=(x_k)\in trf$ if and only if $Vy\in c$ whenever $y=(y_k)\in rf$. Then, we derive by Proposition 1 that $trf^{\beta}=\bigcap_{i=1}^{5} d_i$. □

5. Some Matrix Transformations Related to the Sequence Space *trf*

Dual summability methods are used by many authors, such as Başar [2], Başar and Çolak [3], Lorentz and Zeller [6]. Now, we review to these methods following Başar [2].

Let us suppose that the sequences $x=(x_k)$ and $y=(y_k)$ are connected with (7) and let A-transform of the sequence $x=(x_k)$ be $z=(z_k)$ and B- transform of the sequence $y=(y_k)$ be $p=(p_k)$, i.e.,

$$z_k=(Ax)_k=\sum_k a_{nk}x_k,\ (k\in\mathbb{N}) \tag{29}$$

and

$$p_k=(By)_k=\sum_k b_{nk}y_k,\ (k\in\mathbb{N}). \tag{30}$$

Method B is applied to the $B(r,s,t)$- transform of the sequence $x=(x_k)$ while the method A is directly applied to the terms of the sequence $x=(x_k)$. From here, it is clear that A and B are essentially different [2].

Let us suppose that the matrix product $BB(r,s,t)$ exists. If z_k turns into p_k (or vice versa), under the application of the formal summation by parts, then the methods A and B as in (29) and (30) are named triple dual type matrices. It means that $BB(r,s,t)$ exists and is equal to A. This condition is equivalent to the following equations:

$$b_{nk}=\sum_{j=k}^{\infty}\frac{1}{r}\sum_{m=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-m}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{m}a_{nj}\quad\text{or}\quad a_{nk}=tb_{n,k-2}+sb_{n,k-1}+rb_{nk} \tag{31}$$

for all $n,k\in\mathbb{N}$.

Now we may give the following theorem concerning to the triple dual matrices:

Theorem 5. Let $A=(a_{nk})$ and $E=(e_{nk})$ be the dual matrices of the new type and λ be any given sequence space. Then, $A\in(trf:\lambda)$ if and only if $\{a_{nk}\}_{k\in N}\in trf^{\beta}$ for all $n\in\mathbb{N}$ and $E\in(rf:\lambda)$.

Proof. Let λ be any sequence space and $A=(a_{nk})$ and $E=(e_{nk})$ are triple dual matrices, that is to say that (31) holds. Furthermore, bearing in mind that the spaces trf and rf are isomorphic.

Let $A\in(trf:\lambda)$ and $y=(y_k)\in rf$. Then $EB(r,s,t)$ presents and $(a_{nk})_{k\in N}\in\bigcap_{i=1}^{5}d_i$. It means that $(e_{nk})_{k\in N}\in\ell_1$ for each $n\in\mathbb{N}$. From here, Ey exists and following equation holds;

$$\sum_k e_{nk}y_k=\sum_k a_{nk}x_k \tag{32}$$

for all $n\in\mathbb{N}$, which concluded that $E\in(rf:\lambda)$. On the contrary, let $(a_{nk})_{k\in N}\in trf^{\beta}$ for each $n\in\mathbb{N}$ and $E\in(rf:\lambda)$, and take any $x=(x_k)\in trf$. From here, it is clear that Ax exists. Thus, we attain from the following equality for all $n\in\mathbb{N}$,

$$\sum_{k=0}^{m}a_{nk}x_k=\sum_{k=0}^{m}\sum_{j=k}^{m}b_{jk}^{-1}a_{nj}y_k=\sum_{k=0}^{m}b_{nk}y_k, \tag{33}$$

as $m\to\infty$ that $Ax=Ey$, and it is easy to show that $A\in(trf:\lambda)$. This step completes the proof. □

Theorem 6. Let us assume that the components of the infinite matrices $A=(a_{nk})$ and $E=(e_{nk})$ are connected with the following relation

$$e_{nk}=ta_{n-2,k}+sa_{n-1,k}+ra_{nk}, \tag{34}$$

for all $n,k\in\mathbb{N}$ and λ be any given sequence space. Then, $A\in(\lambda:trf)$ if and only if $E\in(\lambda:rf)$.

Proof. Let us suppose that $x=(x_k)\in\lambda$ and satisfy the following equality for all $n\in\mathbb{N}$:

$$\begin{aligned}\{B(r,s,t)(Ax)\}_n &= t(Ax)_{n-2}+s(Ax)_{n-1}+r(Ax)_n\\ &= t\sum_k a_{n-2,k}x_k + s\sum_k a_{n-1,k}x_k + r\sum_k a_{nk}x_k\\ &= \sum_k (ta_{n-2,k}+sa_{n-1,k}+ra_{nk})x_k\\ &= (Ex)_n.\end{aligned}$$

From here, we can obtain that $Ax\in trf$ if and only if $Ex\in rf$. In this way, we complete the proof.

□

In this section, we characterize the matrix classes $(trf:\ell_\infty),(trf:c),(\ell_\infty:trf)$ and $(c:trf)$ as in the following corollary:

Corollary 2. The following statements hold:

1. $A=(a_{nk})\in(trf:\ell_\infty)$ if and only if $\{a_{nk}\}_{k\in N}\in\{trf\}^\beta$ for all $n\in\mathbb{N}$ and

$$\sup_n\sum_k\left|\frac{1}{r}\sum_{j=k}^{\infty}\sum_{m=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-m}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{m}a_{nj}\right|<\infty. \tag{35}$$

2. $A=(a_{nk})\in(trf:c)$ if and only if $\{a_{nk}\}_{k\in N}\in\{trf\}^\beta$ for all $n\in\mathbb{N}$, (5) and following conditions hold:

$$\lim_n\frac{1}{r}\sum_{j=k}^{\infty}\sum_{m=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-m}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{m}a_{nj}=\alpha_j \text{ for each fixed } k\in\mathbb{N}, \tag{36}$$

$$\lim_n\sum_k\frac{1}{r}\sum_{j=k}^{\infty}\sum_{m=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-m}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{m}a_{nj}=\alpha, \tag{37}$$

$$\lim_n\sum_k\left|\triangle\left(\frac{1}{r}\sum_{j=k}^{\infty}\sum_{m=0}^{j-k}\left(\frac{-s+\sqrt{s^2-4tr}}{2r}\right)^{j-k-m}\left(\frac{-s-\sqrt{s^2-4tr}}{2r}\right)^{m}a_{nj}-\alpha_j\right)\right|=0, \tag{38}$$

$$\sup_n \sum_k \left| \triangle \left(\frac{1}{r} \sum_{j=k}^{\infty} \sum_{m=0}^{j-k} \left(\frac{-s+\sqrt{s^2-4tr}}{2r} \right)^{j-k-m} \left(\frac{-s-\sqrt{s^2-4tr}}{2r} \right)^{m} a_{nj} \right) \right| < \infty. \tag{39}$$

3. $A = (a_{nk}) \in (\ell_\infty : trf)$ if and only if (5) and following statements hold:

$$rf - \lim_n \left(\frac{1}{r} \sum_{j=k}^{\infty} \sum_{m=0}^{j-k} \left(\frac{-s+\sqrt{s^2-4tr}}{2r} \right)^{j-k-m} \left(\frac{-s-\sqrt{s^2-4tr}}{2r} \right)^{m} a_{nj} \right) = \alpha_j, \tag{40}$$

$$\lim_m \sum_k \left| \frac{1}{R_m} \sum_{i=0}^{m} r_i a_{n+i,k} - \alpha_k \right| = 0, \ uni. \ in \ n. \tag{41}$$

4. $A = (a_{nk}) \in (c : trf)$ if and only if (24), (29) and following statement hold:

$$rf - \lim_k \left(\frac{1}{r} \sum_{j=k}^{\infty} \sum_{m=0}^{j-k} \left(\frac{-s+\sqrt{s^2-4tr}}{2r} \right)^{j-k-m} \left(\frac{-s-\sqrt{s^2-4tr}}{2r} \right)^{m} a_{nj} \right) = \alpha. \tag{42}$$

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