Anti-Hausdorff I-Spaces

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Abstract

This is the second paper on I-spaces. Here anti-Hausdorffness has been introduced for I-spaces and many topological theorems related to anti-Hausdorffness have been generalized to I-spaces, as an extension of study of infratopological spaces.

Keywords: I-space, Trivial anti-Hausdorff I-space, Non-trivial anti-Hausdorff I-space, I-Continuous image, Quotient I-space, Irreducible spaces, Infratopological spaces.

1. Introduction

In a previous paper [1] we have introduced I-spaces and studied some of their properties. In this paper we use the terminology of [1]. Some study of these spaces was done previously in ([2], [3], [4], [5]) in less general form. These spaces were called infratopological spaces. Anti-Hausdorff topological spaces and anti-Hausdorff U-spaces were introduced and studied in [6] and [7] respectively. In this paper the concept of anti-Hausdorff I-spaces have been introduced and a few important properties of such spaces have been studied. A number of interesting examples have been constructed to prove non-trivialness of such results.
2. Anti-Hausdorff I-Spaces

We have generalized some results on anti-Hausdorff topological spaces in [6] to I-spaces. We recall that an I-space X is a non-empty set X together with a collection $\mathcal{I}$ of subsets of X such that (i) $\mathcal{I}$ is closed under finite intersections, and (ii) X and $\emptyset$ belong to $\mathcal{I}$.

Definition 2.1. An I-space X with $|X| \geq 2$ is said to be anti-Hausdorff, if for no pair of distinct points x, y in X, there exist I-open sets G and H such that $x \in G$, $y \in H$, $G \cap H = \emptyset$, i.e., if no pair of distinct points can be separated by disjoint I-open sets. Here, $|X|$ denotes the number of elements of X. An anti-Hausdorff I-space which is not a topological space and hence, not a U-space will be called a non-trivial anti-Hausdorff I-space. Otherwise it is called trivial. It is easily seen that anti-Hausdorff I-spaces with 1 or 2 elements are trivial spaces.

Definition 2.2. If $(X, \mathcal{I})$ is an I-space and $\emptyset \neq A \subseteq X$, then $\mathcal{I}_A = \{A \cap G \mid G \in \mathcal{I}\}$ is an I-structure in A. For, $\bigcap_a (A \cap G_a) = A \cap \left( \bigcap_a G_a \right)$ and $\bigcap_a G_a \in \mathcal{I}$. Thus $(A, \mathcal{I}_A)$ is an I-space, and is called an I-subspace of $(X, \mathcal{I})$.

Example 2.1. Let $X = \{a, b, c, d\}$, $\mathcal{I} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, c, d\}\}$. Then $(X, \mathcal{I})$ is non-trivial anti-Hausdorff I-space.

Example 2.2. Let $X = \{a, b, c, d,e\}$ and $\mathcal{I}_1 = \{X, \emptyset, \{a\}, \{a, b\}, \{a, b, d\}\}$, $\mathcal{I}_2 = \{X, \emptyset, \{b\}, \{b, c\}, \{a, b, d\}\}$. Then $(X, \mathcal{I}_1)$ and $(X, \mathcal{I}_2)$ are non-trivial anti-Hausdorff I-spaces.

Example 2.3. Let $X = \mathbb{N}$, $\mathcal{I} = \{X, \emptyset, 2\mathbb{N}, 5\mathbb{N}\}$. Then $(X, \mathcal{I})$ is a non-trivial anti-Hausdorff space.

We introduce the concept of irreducibility as in [6], [7] and [8].

Definition 2.3. An I-space X is said to be irreducible if every pair of non-empty I-open sets in X intersect. Thus an I-space X is irreducible if, for every pair of non-empty I-open sets V, W in X, $V \cap W \neq \emptyset$.

We now prove a theorem which corresponds to Theorem of 4.2 of [6] & Theorem 2.5 of [7].

Theorem 2.1. Let X be an I-space. The following statements are equivalent:

(i) X is anti-Hausdorff,

(ii) X is irreducible,

(iii) The union of two distinct non-empty I-open sets in X is connected,
(iv) Every non-empty I-open set in X is dense in X,

**Proof:** We first prove (i) \(\iff\) (ii).

To prove (i) \(\implies\) (ii) let X be an anti-Hausdorff. If possible suppose that X is not irreducible. Then there exist non-empty I-open sets V and W in X such that \(V \cap W = \emptyset\). Since V and W are non-empty, there exist \(x \in V\) and \(y \in W\). Since \(V \cap W = \emptyset\), \(x \neq y\). X being anti-Hausdorff, this is a contradiction. Therefore X is irreducible.

We now prove (ii) \(\implies\) (i). Let X be irreducible. If possible, let X be not anti-Hausdorff. Then there exist \(x, y \in X\) with \(x \neq y\) and I-open sets V and W in X with \(V \cap W = \emptyset\) and \(x \in V\), \(y \in W\). Since V and W are non-empty, this is a contradiction to the fact that X is irreducible. Hence X is anti-Hausdorff.

To prove (ii) \(\implies\) (iii), Suppose X is irreducible and also suppose (iii) does not hold. Then there exists non-empty distinct I-open sets \(V_1\) and \(V_2\) in X, such that \(V_1 \cup V_2 = V\) (say) is disconnected. Then \(V = V_3 \cup V_4\), for some non-empty I-open sets such that \(V_3 \cap V_4 = \emptyset\). This contradicts (ii) and hence to our hypothesis. Hence the union of distinct non-empty two I-open set in X is connected.

To prove (iii) \(\implies\) (ii), Suppose (iii) holds. If possible, let X be not irreducible. Then there exists non-empty I-open sets \(V_1, V_2\) in X such that \(V_1 \cap V_2 = \emptyset\). Then \(V_1 \cup V_2\) is disconnected. This contradicts (iii). Hence X is irreducible.

We now prove (ii) \(\iff\) (iv). Let X be a irreducible. Let V be a non-empty I-open set in X and let \(x \in X\). Let W be an I-open set in X such that \(x \in W\). Then \(W \neq \emptyset\). Since X is irreducible,

\[V \cap W \neq \emptyset,\] So, \(x \in \overline{V}\). Thus \(X = \overline{V}\), i.e., V is dense in X. Thus (ii) \(\implies\) (iv).

Conversely, suppose that every non-empty I-open set in X is dense in X. Let V and W be two non-empty I-open sets in X and let \(x \in V\). Since \(\overline{W} = X\) and V is a neighborhood of \(x\), \(V \cap W \neq \emptyset\). So X is irreducible.

Therefore (iv) \(\implies\) (ii).

The proof of the theorem is thus complete.

**Theorem 2.2.** Every I-continuous image of an anti-Hausdorff I-space is an anti-Hausdorff I-space.

**Proof:** Let X, Y be two I-spaces where X is anti-Hausdorff I-space. Let f be a I-continuous map of X onto Y. Let \(y_1\) and \(y_2\) be two distinct points of Y, and let \(H_1\) and \(H_2\) be two I-open sets in Y such that \(y_1 \in H_1\), \(y_2 \in H_2\). Since f is onto there exist \(x_1, x_2\) in X such that \(f(x_1) = y_1, f(x_2) = y_2\).

Let \(G_1 = f^{-1}(H_1)\), \(G_2 = f^{-1}(H_2)\). Since f is I-continuous, both \(G_1\) and \(G_2\) are I-open sets. Since X is
Anti-Hausdorff I-Spaces

anti-Hausdorff I-space, \( G_1 \cap G_2 \neq \Phi \). Let \( x \in G_1 \cap G_2 \), then \( f(x) \in H_1 \cap H_2 \). Thus \( H_1 \cap H_2 \neq \Phi \). So, \( Y \) is anti-Hausdorff I-space.

**Definition 2.4.** Let \((X, \mathcal{I})\) be an I-space and \( R \) an equivalence relation on \( X \). The equivalence class for each \( x \in X \) is denoted by \( x \). We define an I-structure \( \mathcal{I} \) on the collection of equivalence classes \( \frac{X}{R} \) of \( X \) with respect to \( R \) as follows. Any subset \( V \) of \( \frac{X}{R} \) will be a member of \( \mathcal{I} \) iff \( \{ x \in X | x \in V \} \in \mathcal{I} \), i.e., for every I-open set \( V \), the collection of equivalence classes of the elements is I-open in \( \frac{X}{R} \), and these are the only I-open members of \( \frac{X}{R} \). That \( \mathcal{I} \) is an I-structure is obvious. This I-structure \( \mathcal{I} \) is called the identification I-structure or the quotient I-structure on \( X \), and \( \left( \frac{X}{R}, \mathcal{I} \right) \) is called the identification I-space or the quotient I-space of \( X \) with respect to \( R \).

**Example 2.4.** Let \( X = \mathbb{R} \), \( \mathcal{I} = \{ \mathbb{R}, \Phi \} \cup \{(n,i, n.(i+1)) \in n \mathbb{Z}, i = 0,1,2,\ldots,9 \} \cup \{(n,j, n.(j + 2)) \in n \mathbb{Z}, j = 0,1,2,\ldots,8 \} \) (Here decimal representation has been used.). Then \((X, \mathcal{I})\) is a non-trivial anti-Hausdorff I-space.

Let \( \rho \) be the relation on \( \mathbb{R} \) given by \( x \rho y \), iff \( x - y \in \mathbb{Z} \). Then \( \rho \) is an equivalence relation on \( \mathbb{R} \), and for each \( x \in \mathbb{R} \), \( \bar{x} = x + \mathbb{Z} \), the coset of \( \mathbb{Z} \) in \( \mathbb{R} \), both \( \mathbb{Z} \) and \( \mathbb{R} \), being regarded as additive groups.

The quotient I-structure \( \mathcal{I} \) on \( \frac{\mathbb{R}}{\rho} \) is given by

\[
\mathcal{I} = \left\{ \frac{\mathbb{R}}{\rho}, \Phi \right\} \cup \{(m,i, m.(i+1)) + Z | m \in \mathbb{Z} \} \cup \{(n,j, n.(j + 2)) + Z | n \in \mathbb{Z} \},
\]

where for integers \( m \) and \( n \), \( (m,i, m.(i+1)) + Z = \{(m + r).i, (m + r).(i + 1)) \mid r \in \mathbb{Z} \}, \text{ and } (n,j, n.(j + 2)) + \mathbb{Z} = \{(n + s).j, (n + s).(j + 2)) \mid s \in \mathbb{Z} \}.

Hence \( \frac{\mathbb{R}}{\rho} \) is a non-trivial anti-Hausdorff I-space.

**Corollary 2.1.** If \( X \) is an anti-Hausdorff I-space and \( R \) is an equivalence relation on \( X \), then the quotient I-space \( \frac{X}{R} \) is anti-Hausdorff I-space.
Proof: It follows from the definition of quotient I-space that the map \( f: X \to \frac{X}{R} \) given by \( f(x) = \text{cls } x \) is continuous and onto. The corollary is then follows from Theorem 2.2.

**Theorem 2.3.** A U-subspace of a non-trivial anti-Hausdorff I-space need not be U-anti-Hausdorff.

**Proof:** Let us consider the I-space \( (X, \mathcal{I}) \), where \( X = \{a, b, c, d\} \) and \( \mathcal{I} = \{X, \Phi, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a\}\} \). Then \( (X, \mathcal{I}) \) is a non-trivial anti-Hausdorff I-space, since there is no pair of disjoint non-empty I-open sets in \( X \). Now let \( Y = \{b, c, d\} \). Then as a subspace of \( X \), \( Y \) has the I-structure, \( \mathcal{I} = \{Y, \Phi, \{b\}, \{c\}, \{c, d\}\} \). Obviously, \( Y \) is not anti-Hausdorff I-space.

**Theorem 2.4.** If \( A \) and \( B \) two non-trivial anti-Hausdorff I-subspaces of an I-space \( X \), then the subspace \( A \cap B \) need not be non-trivial anti-Hausdorff I-space.

**Proof:** Let \( X = \{a, b, c, d, e, f\} \), \( \mathcal{I} = \{X, \Phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{b, c, d, e\}, \{b, c, d, f\}\} \). Clearly \( (X, \mathcal{I}) \) is a non-trivial I-space. Let \( A = \{a, b, c, d, f\} \) and \( B = \{a, b, d, f\} \). Then \( A \) and \( B \) are I-subspaces of \( X \) with \( \mathcal{I}_A = \{A, \Phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{b, c, d, f\}\} \), \( \mathcal{I}_B = \{B, \Phi, \{b\}, \{a, b\}, \{b, d\}, \{b, d, f\}\} \). Clearly both \( A \) and \( B \) are non-trivial anti-Hausdorff I-subspaces of \( X \). Now \( A \cap B = \{a, d, f\} \) and \( \mathcal{I}_{A \cap B} = \{A \cap B, \Phi, \{a\}, \{d\}, \{d, f\}\} \). Then \( A \cap B \) is a non-trivial I-space which is not anti-Hausdorff.

The following provides another example which proves the truth of theorem 2.4.

**Example 2.5.** Let \( X = \{a, b, c, d, e, f\} \), \( \mathcal{I} = \{X, \Phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{b, c, d, e\}, \{b, c, d, f\}\} \). Clearly \( (X, \mathcal{I}) \) is a non-trivial I-space. Let \( A = \{a, b, c, d, e\} \) and \( B = \{a, b, c, d, f\} \). Then \( A \) and \( B \) are I-subspaces of \( X \) with \( \mathcal{I}_A = \{A, \Phi, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{b, c, d, e\}\} \), \( \mathcal{I}_B = \{B, \Phi, \{b\}, \{a, b\}, \{b, c\}, \{b, c, d\}, \{b, c, d, f\}\} \). Clearly both \( A \) and \( B \) are non-trivial anti-Hausdorff I-subspaces of \( X \). Now \( A \cap B = \{a, b, c, d\} \) and \( \mathcal{I}_{A \cap B} = \{A \cap B, \Phi, \{a\}, \{b\}, \{c, d\}\} \). Then \( A \cap B \) is a trivial anti-Hausdorff I-space. Thus \( A \cap B \) is not a non-trivial anti-Hausdorff I-space.

In the situation of Theorem-2.4, it is also possible that \( A \cap B \) is a non-trivial anti-Hausdorff I-space as is shown by the following example.

**Example 2.6.** Let \( X = \{a, b, c, d, e, f, g\} \), \( \mathcal{I} = \{X, \Phi, \{a, b, c\}, \{a, b, c, d\}, \{b, c, d, e\}, \{b, c, d, f\}\} \). Clearly \( (X, \mathcal{I}) \) is a non-trivial I-space. Let \( A = \{a, b, c, d, e\} \) and \( B = \{a, b, c, d, f\} \). Then \( A \) and \( B \) are I-subspaces of \( X \) with \( \mathcal{I}_A = \{A, \Phi, \{a, b, c\}, \{a, b, c, d\}, \{b, c, d\}, \{b, c, d, e\}\} \), \( \mathcal{I}_B = \{B, \Phi, \{a, b, c, d\}, \{b, c, d\}, \{b, c, d, e\}, \{b, c, d, f\}\} \). Clearly both \( A \) and \( B \) are non-trivial anti-Hausdorff I-subspaces of \( X \). Now \( A \cap B = \{a, b, c, d\} \) and \( \mathcal{I}_{A \cap B} = \{A \cap B, \Phi, \{a\}, \{b\}, \{c\}, \{d\}\} \). Then \( A \cap B \) is a non-trivial anti-Hausdorff I-space.

**Remark 2.1.** If \( A \) and \( B \) are two non-trivial subspaces of a non-trivial I-space \( X \), then the subspace \( A \cap B \) may be non-trivial anti-Hausdorff I-space even if neither \( A \) nor \( B \) is so.
Let $X = \mathbb{R}$, $\mathcal{I} = \{\mathbb{R}, \emptyset, (a, b) | a, b \in \mathbb{R}, a < b\}$. Clearly $(X, \mathcal{I})$ is a non-trivial I-space.

Let $A = (2, 5) \cup \mathbb{Q}^c$ and $B = \left(\sqrt{7}, \sqrt{29}\right) \cup \mathbb{Q}$, where $\mathbb{Q}^c = \mathbb{R} - \mathbb{Q}$. Then the I-structure $\mathcal{I}_A$ and $\mathcal{I}_B$ on $A$ and $B$ respectively are $\mathcal{I}_A = \{A, \emptyset\} \cup \{(a, b) | a, b \in \mathbb{R}, 2 \leq a < b \leq 5\} \cup \{(c, d) \cap \mathbb{Q}^c | c, d \in \mathbb{R}\}$.

Since $(3, 4) \cap (5, 6) \cap \mathbb{Q}^c = \emptyset$, $(A, \mathcal{I}_A)$ is not anti-Hausdorff I-subspace.

$\mathcal{I}_B = \{B, \emptyset\} \cup \{(a, b) | a, b \in \mathbb{R}, \sqrt{7} < a < b < \sqrt{29}\} \cup \{(c, d) \cap \mathbb{Q} | c, d \in \mathbb{R}\}$. Since $(\sqrt{7}, \sqrt{29}) \cap (6, 7) \cap \mathbb{Q} = \emptyset$, $(B, \mathcal{I}_B)$ is not anti-Hausdorff I-subspace. Thus both $A$ and $B$ are non-trivial subspaces of an I-space $X$, neither of which is anti-Hausdorff.

\[ A \cap B = \left(\sqrt{7}, 5\right) \cup ((2, 5) \cap \mathbb{Q}) \cup ((\sqrt{7}, \sqrt{29}) \cap \mathbb{Q}^c) \] and $\mathcal{I}_A \cap \mathcal{I}_B = \{A \cap B, \emptyset\} \cup \{(a, b) | a, b \in \mathbb{R}, a < b\} \cup ((2, 5) \cap \mathbb{Q}) \cap (a, b) | a, b \in \mathbb{R}, a < b\} \cup ((\sqrt{7}, \sqrt{29}) \cap \mathbb{Q}^c) \cap (a, b) | a, b \in \mathbb{R}, a < b\}$

Thus $A \cap B$ is a non-trivial anti-Hausdorff I-space.

As in Theorem 2.9 of [6] and Theorem 2.3 of [7], we have

Theorem 2.5. Let $A_1$ and $A_2$ be two anti-Hausdorff I-spaces with I-structures $\mathcal{I}_1$ and $\mathcal{I}_2$ respectively.

Then $(A_1 \cup A_2, \left\{\mathcal{I}_1 \cup \mathcal{I}_2\right\})$ need not be anti-Hausdorff I-space. Here $\left\{\mathcal{I}_1 \cup \mathcal{I}_2\right\}$ is the I-structure generated by $\mathcal{I}_1 \cup \mathcal{I}_2$ in $A_1 \cup A_2$.

**Proof:** Let $A_1 = \{a, c, d, e\}$, $\mathcal{I}_1 = \{A_1, \emptyset, \{a, d\}, \{a, e\}, \{a\}\}$, and $A_2 = \{a, c, d, f\}$, $\mathcal{I}_2 = \{A_2, \emptyset, \{c, d\}, \{c, f\}, \{c\}\}$. Then $(A_1, \mathcal{I}_1)$ and $(A_2, \mathcal{I}_2)$ are non-trivial anti-Hausdorff I-spaces.

Then $A = A_1 \cup A_2 = \{a, c, d, e, f\}$. Let $\mathcal{I}$ be the I-structure on $A$ generated by $\mathcal{I}_1 \cup \mathcal{I}_2$, i.e., $\mathcal{I} = \{A, A_1, A_2, \emptyset, \{a\}, \{c, d\}, \{c, f\}, \{a, d\}, \{a, e\}\}$. So, in $(X, \mathcal{I})$, $a \in \{a\}, d \in \{d\}$ with $\{a\}, \{d\} \in \mathcal{I}$ and $\{a\} \cap \{d\} = \emptyset$.

Hence $(X, \mathcal{I})$ is not an anti-Hausdorff I-space.

However, we also have

Theorem 2.6. If $A$ and $B$ are two non-trivial anti-Hausdorff I-subspaces of an I-space $X$, then $A \cup B$ may be non-trivial anti-Hausdorff.
Proof: Let X = \{a, b, c, d, e, f, g\}, \mathcal{I} = \{X, \emptyset, \{a, b, c\}, \{b, c, e, f\}, \{b, c\}, \{a, b\}, \{a, b, d\}, \{b\}\}. Clearly (X, \mathcal{I}) is a non-trivial anti-Hausdorff I-space. Let A = \{a, b, d, e, f\}, B = \{b, c, d, g\}. Then A and B are I-subspaces of X with \mathcal{I}_A = \{A, \emptyset, \{a, b\}, \{b, e, f\}, \{b\}\}, \mathcal{I}_B = \{B, \emptyset, \{c\}, \{a, d\}\}. Clearly, both A and B are non-trivial anti-Hausdorff I-subspaces of X. Now A \cup B = \{a, b, c, d, e, f, g\} and \mathcal{I}_{A \cup B} = \{A \cup B, \emptyset, \{a, b\}, \{b, e, f\}, \{b\}\}. Hence, A \cup B = X too, is a non-trivial anti-Hausdorff I-space.

The following theorems prove the truth of the various possibilities for A \cup B, where A, B are I-subspaces of X.

Theorem 2.7. If A is a non-trivial anti-Hausdorff I-subspace and B is a non-trivial non anti-Hausdorff I-subspace of X, then A \cup B may be a non-trivial anti-Hausdorff I-subspace of X.

Proof: Let X = \{a, b, c, d, e, f, g\}, \mathcal{I} = \{X, \emptyset, \{a, b, c\}, \{b, c, e, f\}, \{b, c\}, \{a, b\}, \{a, b, d\}, \{b\}\}. Clearly (X, \mathcal{I}) is a non-trivial anti-Hausdorff I-space. Let A = \{a, b, d, e, f\}, B = \{c, d, g\}. Then A and B are I-subspaces of X with \mathcal{I}_A = \{A, \emptyset, \{a\}, \{e, f\}, \{a, d\}\}, \mathcal{I}_B = \{B, \emptyset, \{c\}, \{f\}\}. Clearly, A is a non-trivial anti-Hausdorff I-subspace and B is a non-trivial non anti-Hausdorff I-subspace of X. Now A \cup B = \{a, b, c, d, e, f, g\} and \mathcal{I}_{A \cup B} = \{A \cup B, \emptyset, \{a\}, \{e, f\}, \{a, d\}\}. Here A \cup B is a non-trivial anti-Hausdorff I-subspace of X.

Theorem 2.8. If A and B are non-trivial non anti-Hausdorff I-subspaces of an I-space X, then A \cup B may be a non-trivial anti-Hausdorff I-subspace of X.

Proof: Let X = \{a, b, c, d, e, f, g\}, \mathcal{I} = \{X, \emptyset, \{a, b, c\}, \{b, c, e, f\}, \{b, c\}, \{a, b\}, \{a, b, d\}, \{b\}\}. Clearly, (X, \mathcal{I}) is a non-trivial anti-Hausdorff I-space. Let A = \{a, b, c, d, e, f\}, B = \{c, d, g\}. Then A and B are I-subspaces of X with \mathcal{I}_A = \{A, \emptyset, \{a\}, \{e, f\}, \{a, d\}\}, \mathcal{I}_B = \{B, \emptyset, \{c\}, \{f\}\}. Clearly, A and B are non-trivial non anti-Hausdorff I-subspaces of X. Now A \cup B = \{a, b, c, d, e, f, g\} and \mathcal{I}_{A \cup B} = \{A \cup B, \emptyset, \{a\}, \{e, f\}, \{a, d\}\}. Here A \cup B is a non-trivial non anti-Hausdorff I-subspace of X.

Theorem 2.9. If A and B are non-trivial non anti-Hausdorff I-subspace of an I-space X, then A \cup B
may be a non-trivial anti-Hausdorff I-subspace of X.

The following example proves the truth of the theorem.

**Proof:** Let $X = \mathbb{R}$, $\mathcal{I} = \{\mathbb{R}, \emptyset, [2,4], [3,5], [3,4] \cup [7,10]\}$. Then $X$ is a non-trivial anti-Hausdorff I-space. Let $A = ([2,3] \cap \mathbb{Q}^c) \cup ([3,4] \cap \mathbb{Q}) \cup ([4,5] \cap \mathbb{Q}^c) \cup [7,10]$. Then $\mathcal{I}_A = \{A, \emptyset, ([2,3] \cap \mathbb{Q}^c), ([3,4] \cap \mathbb{Q}), ([4,5] \cap \mathbb{Q}^c)\}$. $A$ is non-trivial non-anti-Hausdorff I-subspace of an I-space $X$. Let $B = ([2,3] \cap \mathbb{Q}) \cup ([3,4] \cap \mathbb{Q}^c) \cup ([4,5] \cap \mathbb{Q})$, $\mathcal{I}_B = \{B, \emptyset, ([2,3] \cap \mathbb{Q}), ([3,4] \cap \mathbb{Q}^c), ([4,5] \cap \mathbb{Q})\}$. $B$ is non-trivial non-anti-Hausdorff I-subspace of an I-space $X$.

Now $A \cup B = [2, 3) \cup [3,4] \cup (4,5] \cup [7,10] = [2,5] \cup [7,10]$. Therefore $\mathcal{I}_{A \cup B} = \{A \cup B, \emptyset, [2,4], [3,5], [3,4]\}$.

Thus $A \cup B$ is a non-trivial anti-Hausdorff I-subspace of $X$.

**Theorem 2.10.** If $A$ is a non-trivial anti-Hausdorff I-subspace and $B$ is non-trivial non anti-Hausdorff subspace of an I-space $X$, then $A \cup B$ may be a non-trivial non anti- Hausdorff I-subspace of $X$.

**Proof:** Let $X = \mathbb{R}$, $\mathcal{I} = \{\mathbb{R}, \emptyset, (1,5), [2,3), (1,2], [2,4], (-3, -2), (-4, -3), \{2\}\}$. Then $X$ is non-trivial non anti-Hausdorff I-space. Let $A = [1,4)$, then $\mathcal{I}_A = \{A, \emptyset, (1,4), [2,3), (1,2], [2,4), \{2\}\}$. $A$ is non-trivial anti-Hausdorff I-subspace of an I-space $X$.

Let $B = (-4, -1)$, $\mathcal{I}_B = \{B, \emptyset, (-3, -2), (-4, -3)\}$. $B$ is non-trivial non anti-Hausdorff I-subspace of an I-space $X$.

Now $A \cup B = [(1, 4) \cup (- 4, -1)]$, $\mathcal{I}_{A \cup B} = \{A \cup B, \emptyset, (1,4), [2,3), (1,2], [2,4), \{2\}, (-3, -2), (-4, -3)\}$.

Here $A \cup B$ is a non-trivial non anti-Hausdorff I-subspace of $X$.

**Theorem 2.11.** If $A$ and $B$ two non-trivial anti-Hausdorff I-subspaces of a I-space $X$, then $A \cup B$ may be a non-trivial non anti-Hausdorff I-subspace of $X$.

**Proof:** Let $X = \mathbb{R}$, $\mathcal{I} = \{\mathbb{R}, \emptyset, [1,5), [2,3), (1,2], [2,4), \{2\}, [-3, -2), [-2, -1), \{-2\}\}$. Then $X$ is non-trivial not anti-Hausdorff I- space. Let $A = [1,3)$, then $\mathcal{I}_A = \{A, \emptyset, (2,3), (1,2], \{2\}\}$. $A$ is non-trivial anti-Hausdorff I-subspace of an I-space $X$.

Let $B = [-4, -1)$, $\mathcal{I}_B = \{B, \emptyset, [-3, -2), [-2,-1), \{-2\}\}$. $B$ is non-trivial anti-Hausdorff I-subspace of an I-space $X$.

Now $A \cup B = [(1, 3) \cup [-4, -1)]$, $\mathcal{I}_{A \cup B} = \{A \cup B, \emptyset, (1,3), (2,3), (1,2], \{2\}, [-3, -2), [-2, -1), \{-2\}\}$.

Here $A \cup B$ is a non-trivial non anti-Hausdorff I-subspace of $X$. Anti-Hausdorff I-Spaces


References


