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# Constant Proportion Portfolio Insurance Strategies in Hybrid Markets

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## Abstract

Financial decisions are made under the state of indeterminacy. Randomness and fuzziness are two basic forms of indeterminacy. Probability theory (Kolmogorov, 1933) models randomness and fuzzy set theory (Zadeh, 1965) deals with fuzziness. However, in some cases, randomness and fuzziness appear simultaneously in a mathematical system. In order to deal with mathematical systems that contain both fuzziness and randomness, chance theory has been developed. Portfolio insurance refers to investment strategies which guarantee that the portfolio value at maturity or at any time before maturity will not go below a stated lower bound (also known as the floor), usually fixed as a fraction of the initial principal investment (Cont and Tankov, 2007). Constant proportion portfolio insurance (CPPI) is a popular example of portfolio insurance techniques. Various research papers have examined CPPI strategies based on probability theory (for example, Neftci, 2008 and Cont and Tankov, 2007) and credibility theory (see, for instance, Matenda, 2016). This study aims to analyse the mechanics of CPPI strategies in hybrid markets. Hybrid markets are markets in which asset prices are driven by hybrid processes. Basically, hybrid processes model both randomness and fuzziness. Assuming diffusion models with continuous trading, CPPI strategies are not exposed to gap risk. However, in reality, CPPI techniques are exposed to gap risk which needs to be analytically quantified. The multiplier value, $m$, and the CPPI-insured portfolio value, $V_t$, are positively correlated. A direct relationship between the CPPI-insured portfolio risk of loss and the multiplier value has been substantiated. The research paper constructs a strong foundation for the calculation of the multiplier value in accordance with the risk tolerance of the investors. This peace of research work also unfolds the basis for the analytical quantification of gap risk for CPPI strategies when asset price processes evolve as hybrid processes with jumps. The study is the first peace of research work to analyse CPPI strategies in hybrid markets.

*Keywords*: Portfolio insurance, indeterminacy, randomness, fuzziness, chance theory, hybrid processes, hybrid markets, gap risk, multiplier value.

# 1. Introduction

Financial decisions are made under conditions of indeterminacy. Peng (2013) defines indeterminacy as phenomena whose outcomes cannot precisely be determined in advance. Indeterminacy is also conceptualised as a condition of events' outcomes being unpredictable in advance (Matenda, Chikodza and Gumbo, 2015). Examples of indeterminate phenomena include stock price, heavy loss and tossing a dice. Indeterminacy gives rise to the riskiness of a business venture. Empirical and theoretical evidence has indicated that risk is inherent in every investment undertaken. In business and finance risk management is an important practice.

Randomness and fuzziness are two basic forms of indeterminacy. Probability theory (Kolmogorov, 1933) and fuzzy set theory (Zadeh, 1965) deal with randomness and fuzziness, respectively. Any phenomenon which can be quantified by a probability measure is called randomness (Liu, 2012). Matenda, Chikodza and Gumbo (2015) propound that randomness is an attribute of anything which can be described by probability. On the other hand, fuzziness is defined as any phenomenon which can be described by a credibility measure. Jiwo and Chikodza (2015) propose that fuzziness is a concept that describes processes or events whose measurement is intrinsically dim and imperfect.

Pascal and Fermat pioneered the study of probability in 1654 and subsequently, Kolmogorov (1933) introduced the foundation of probability theory. Probability theory is an axiomatic branch of pure mathematics for studying the behaviour of dynamic random systems which is based on normality, non-negativity and additivity axioms (Liu, 2007). Conceptually, probability theory models frequency and is used when a large volume of historical data is available. Stochastic processes model randomness. Liu (2015) defines a stochastic process as a sequence of random variables indexed by time. A Brownian motion, pioneered by Robert Brown in 1827 and improved by Einstein in 1905, is one of the popular and widely used stochastic processes. Norbert Wiener further revised a Brownian motion in 1923. As a result, a Brownian motion can be called a Wiener process or a Wiener-Einstein process.

The application of probability theory in the discipline of finance, started in 1900 by Bachelier, led to the emergence of stochastic finance theory. A Brownian motion was first introduced in the field of finance by Bachelier in 1900. Bachelier proposed that stock prices follow a Brownian motion despite the fact that a Brownian motion predicts negative stock prices. Samuelson (1965) suggests that stock prices follow a geometric Brownian motion without taking into account the time value of money, among other issues. Subsequently, Black and Scholes (1973) and Merton (1973) proffer that stock prices follow a geometric Brownian motion. The famous Black-Scholes model was suggested by Black and Scholes (1973) under the assumption that stock prices follow a geometric Brownian motion.

Based on a Brownian motion, Ito (1944) introduces stochastic calculus which is a branch of mathematics that deals with the integration and differentiation of functions of stochastic processes. Several differential equations in stochastic calculus are driven by a Brownian motion. Liu (2008) defines a stochastic differential equation as a differential equation driven by a Brownian motion. Ito integral was initiated by Ito (1949). Kunita and Watanabe (1967) and Kallenberg (1997) are some of the authors who have contributed a lot in stochastic calculus.

Zadeh (1965) suggests the fuzzy set via membership function which can be estimated by experienced experts. Fuzzy set theory is basically used when historical data is not available. In order to measure a fuzzy event, Liu and Liu (2002) propose a credibility measure. A sufficient and necessary condition for a credibility measure was suggested by Li and Liu (2006). Liu (2004) pioneered credibility theory and Liu (2007) further revised it. Credibility theory is an axiomatic branch of pure mathematics that models the behaviour of dynamic fuzzy phenomena which is based on normality, monotonicity, self-duality and maximality axioms.

Liu (2008) introduces a notion of a fuzzy process in order to model dynamic fuzzy phenomena which vary with time. A fuzzy process is a family of fuzzy variables which vary with time. Liu (2008) proposes a Liu process and, Zhao and Liu (2003) suggest a renewal process. A Liu process and a renewal process are two popular examples of fuzzy processes. Peng (2008) propounds that a Liu process and a geometric Liu process are the fuzzy counterparts of a Brownian motion and a geometric Brownian motion, respectively.

Based on a Liu process, Liu (2008) pioneered fuzzy calculus. Fuzzy calculus is a branch of pure mathematics that deals with the integration and differentiation of functions of fuzzy processes. A Liu integral which is a fuzzy integral with respect to a Liu process was initiated by Liu (2008). Liu (2008) introduces a Liu formula in order to differentiate functions of a Liu process. The existence and uniqueness theorem for homogeneous fuzzy differential equations was proposed by You (2008). A fuzzy differential equation is a differential equation which is driven by a Liu process. Fuzzy calculus with jump processes was suggested by Liu (2008) in order to deal with jumps in fuzzy processes. A fuzzy differential equation with jumps is a differential equation driven by both a Liu process and a renewal process.

Fuzzy finance theory has its roots in fuzzy mathematics. Liu (2008) is the first person to introduce fuzzy calculus and fuzzy differential equations in the discipline of finance under the assumption that stock prices follow a geometric Liu process. A fuzzy stock model named Liu's stock model, which is regarded as a fuzzy counterpart of a Black-Scholes model, was developed by Liu (2008). In order to incorporate asset price shocks into a stock model, Liu (2008) once again constructs a fuzzy stock model with jumps. For more technical and detailed expositions on the application of fuzzy calculus in finance the reader is referred to, among other sources, Liu (2008), Qin and Li (2008), Gao (2008), Peng (2008), Yoshida et al. (2006) and Yoshida (2003).

However, in some instances, randomness and fuzziness simultaneously appear in a mathematical system. A fuzzy random variable was proposed by Kwakernaak (1979, 1978), as a random element taking "fuzzy variable" values, in order to describe systems which exhibit both randomness and fuzziness. Liu (2002) suggests a random fuzzy variable as a fuzzy element taking "random variable" values. A hybrid variable, introduced by Liu (2006), is a tool to describe the quantities with fuzziness and randomness. A random fuzzy variable and a fuzzy random variable are cases of hybrid variables. A hybrid variable is defined as a measurable function from a chance space to the set of real numbers. Li and Liu (2008) introduces a chance measure in order to measure hybrid events. Furthermore, a tool called chance theory was developed in order to study the behaviour of dynamic phenomena with fuzziness and randomness. Chance theory is defined as a branch of pure mathematics which models dynamic phenomena with randomness and fuzziness. Generally speaking, chance theory is regarded as a counterpart of probability theory and fuzzy theory which deals with the dynamics of phenomena with fuzziness and randomness.

In order to deal with the evolution of dynamic hybrid phenomena indexed by time, Liu (2008) suggests a hybrid process, a hybrid integral and a hybrid differential equation. Liu (2008) introduces a D process (known as a Wiener-Liu process) and a hybrid renewal process which are two common types of hybrid processes. A Wiener-Liu process is a hybrid process which is a counterpart of a Brownian motion in stochastic processes and Liu process in fuzzy processes. In order to model jumps in hybrid processes, a hybrid renewal process has been developed. Based on a hybrid process, hybrid calculus was initiated by Liu (2008). Hybrid calculus is a branch of mathematics that deals with the integration and differentiation of functions of hybrid processes. Ito-Liu formula is a counterpart of the Ito formula in probability theory and Liu formula in fuzzy set theory. Moreover, Ito-Liu integral is regarded as the counterpart of Ito integral in probability theory and Liu integral in fuzzy set theory.

Liu (2008) introduces a hybrid differential equation which is defined as a differential equation driven by a Wiener-Liu process. A basic hybrid stock model with both randomness and fuzziness was initiated by Liu (2008). Furthermore, in order to model jumps in hybrid process, Liu (2008) pioneered a hybrid renewal process and proposed a hybrid differential equation with jumps. A hybrid differential equation with jumps is a differential equation driven by a Wiener-Liu process and a hybrid renewal process. Liu (2008) also initiated a hybrid stock model with jumps. Conclusively, stochastic finance theory is based on the assumption that stock prices follow a geometric Brownian motion whilst fuzzy finance theory suggests that stock prices are described by a geometric Liu process. On the other hand, Liu (2008) proposes an alternative assumption that stock prices follow a geometric Wiener-Liu process.

## CPPI

Portfolio insurance refers to investment management techniques which guarantee that the portfolio value at maturity or at any time before maturity will not go below a stated lower bound, usually fixed as a fraction of the initial principal investment (Cont and Tankov, 2007). These investment management techniques limit the downside risk of the portfolio whilst maintaining that portfolio's upside potential. Portfolio insurance is one interesting and popular area of dynamic asset allocation (DAA) techniques. Investment management strategies which shift portfolios between risky and risk-free asset classes throughout the investment period in response to investor demands and market developments are called DAA techniques (Trippi and Harriff, 1991).

The evolution in technology, recent breakthroughs in financial engineering and the availability of high frequency data increase the riskiness of world financial markets. On the other side, investors are becoming more and more risk averse with each passing day. Portfolio insurance is very critical in empirical finance and risk management. In a seminar paper, Leland (1980) proposes that the following two types of investors should buy portfolio insurance:

- investors that have expectations levels that are above average and average risk tolerance levels, and
- investors that have risk tolerance levels that increase with wealth faster than average and average expectations levels.

Option based portfolio insurance (OBPI) is one of the several popular portfolio insurance strategies. An OBPI strategy combines an exposure in the risky asset with a put option on that risky asset. This study focuses on CPPI which is also one of the prominent examples of portfolio insurance techniques. The concept of CPPI does not use options. Therefore, CPPI is an easy to implement technique. Attractive features of CPPI techniques include principal protection capacity, simplicity, favourable regulatory behaviour and flexibility. The notion of CPPI was pioneered by Perold (1986) for fixed income instruments and, Black and Jones (1987) for equity instruments.

A CPPI technique allows an investor to limit the downside risk of a portfolio whilst maintaining the upside potential of that same portfolio. Compared to the unprotected portfolio, the upside potential of a CPPI protected portfolio is reduced (Cont and Tankov, 2007). Adopting a CPPI strategy allows the investor to influence the future returns of the investment through continuous dynamic rebalancing of the portfolio (Schied, 2014). When financial markets are bullish, an investor shifts the portfolio towards a risky asset at the expense of a riskless asset. The investor, therefore, reap higher returns from a risky asset. However, when financial markets are bearish, an investor shifts the portfolio towards a risk-free asset at the expense of a risky asset. The continuous portfolio rebalancing exercise enables an investor to protect the initial principal investment.

The CPPI strategies are negative gamma products. When markets are rising the investor buys a risky asset at the expense of a default-free bond and the converse is true. The common economic practice is for an investor to buy an asset when the price is low and sell it when the price is high. In CPPI strategies the investor buys an asset when the price is high and sells it when the price is low.

At the end of the investment tenure, $T$, a CPPI strategy guarantees the initial principal investment, $F$. A CPPI strategy is based on the concept of the cushion, $U_t$, which is defined as the difference between the total portfolio value, $V_t$, and the portfolio lower bound (floor), $P_t$. That is, $U_t = V_t - P_t$. The portfolio lower bound is a well determined value at which the total portfolio value is not allowed to fall below. Portfolio value is the total market value of a risky asset and a default-free bond.

Part of the portfolio is invested in a risky asset (for example, a financial index such as the FTSE All-Share or a portfolio of stocks) and the remainder is placed in a riskless bond (usually a risk-free government bond) with maturity $T$ and nominal value $F$ (Cont and Tankov, 2007). An amount placed in a risky asset, $Y_t$, is proportional to the cushion. Mathematically, $Y_t = mU_t$, where $m$ is the multiplier value known as the participation rate. The participation rate indicates the leverage level the investor is prepared to tolerate. Conceptually, the parameter $m$ is assumed to be greater than 1 and constant throughout the investment tenure. To determine the amount invested in a zero-coupon bond, $A_t$, $Y_t$ is deducted from $V_t$, that is, $A_t = V_t - Y_t$. Generally, in a CPPI strategy, the following conditions hold

(i). if $V_t > P_t$, $Y_t$ is placed in a risky asset and $A_t$ is invested in a riskless asset.

(ii). if $P_t \geq V_t$, all money is invested in a default-free bond to protect the initial principal investment.

(iii). at $t = 0$, $V_0 > P_0$.

(iv). If $Y_t > V_t$, $Y_t$ is invested in a risky asset and the remaining $Y_t - V_t$ is borrowed. However, if $Y_t < V_t$, an investment in a risky asset does not need any additional borrowing.

The remainder of the research paper is organised as follows: Section 2 presents the Preliminaries and Problem Formulation; a detailed analysis of the mechanics of CPPI in Hybrid Markets is performed in Section 3 and lastly, Conclusions are specified in Section 4.

## 2. Preliminaries and Problem Formulation

From a mathematical perspective, the study assumes a chance space $\{\Theta, \mathcal{P}, \text{Cr}\} \times \{\Omega, \mathcal{A}, \Pr\}$ and filtration, $\{\mathcal{F}_t\}_{t \in [0,T]}$, generated by a standard one-dimensional Liu process $\{C_t\}_{t \geq 0}$, and a one-dimensional Brownian motion $\{W_t\}_{t \geq 0}$, which are specified in the models.

**Definition 2.1** (Liu, 2015) Suppose $\Omega$ is a non-empty set and $\mathcal{A}$ is a $\sigma$-algebra over $\Omega$. Each element A in $\mathcal{A}$ is an event. A probability measure is defined as a set function Pr: $\mathcal{A} \to [0,1]$ which satisfies the following three axioms:

- **Axiom 1:** (Normality) $\Pr\{\Omega\} = 1$ for the universal set $\Omega$;
- **Axiom 2:** (Non-negativity) $\Pr\{A\} \geq 0$ for any event A;
- **Axiom 3:** (Additivity) For every countable sequence of mutually disjoint events $A_1, A_2, \ldots$, we have

$$\Pr\{\bigcup_{i=1}^{\infty} A_i\} = \sum_{i=1}^{\infty} \Pr\{A_i\}. \tag{2.1}$$

**Definition 2.2** (Liu, 2015) A random variable is defined as a function $\xi$ from a probability space ($\Omega$, $\mathcal{A}$, Pr) to the set of real numbers such that for any Borel set B of real numbers, $\{\xi \in B\}$ is an event.

**Definition 2.3** (Liu, 2015) Suppose ($\Omega$, $\mathcal{A}$, Pr) is a probability space and $T$ is an index set. A stochastic process is defined as a measurable function $X_t(\omega)$ from $T \times (\Omega, \mathcal{A}, \text{P})$ to the set of real numbers such that for any Borel set *B* of real numbers at each time $t$, $\{X_t \in B\}$ is an event. Generally speaking, a stochastic process is a sequence of random variables indexed by time or space.

**Definition 2.4** (Liu, 2015) A stochastic process $B_t$ is a standard Brownian motion if

- $B_0 = 0$ and almost all sample paths are continuous,
- $B_t$ has stationary and independent increments,
- every increment $B_{s+t} - B_s$ is a normal random variable with expected value $0$ and variance $t$.

**Definition 2.5** (Liu, 2008) Suppose $\Theta$ is a non-empty set and $\mathcal{P}$ is a power set of $\Theta$. Every element A in $\mathcal{P}$ is an event. A credibility measure is a set function Cr: $\mathcal{P} \to [0,1]$ which satisfies the four axioms below:

- **Axiom 1:** (Normality) $\text{Cr}\{\Theta\} = 1$;
- **Axiom 2:** (Monotonicity) $\text{Cr}\{A\} \le \text{Cr}\{B\}$ if $A \subset B$;
- **Axiom 3:** (Self-Duality) $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$ for any $A \in \mathcal{P}$;
- **Axiom 4:** (Maximality) $\text{Cr}\{\cup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any events $\{A_i\}$ with $\sup_i$ $\text{Cr}\{A_i\} < 0.5$.

**Definition 2.6** (Liu, 2008) A fuzzy variable is a measurable function from a credibility space ($\Theta, P$, Cr) to the set of real numbers.

**Definition 2.7** (Liu, 2008) Suppose $T$ is an index set and ($\Theta$, $\mathcal{P}$, Cr) is a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.

**Definition 2.8** (Liu, 2008) A fuzzy process $C_t$ is a standard Liu process if

- $C_0 = 0$,
- $C_t$ has stationary and independent increments,
- every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value $et$ and variance $\sigma^2 t^2$, whose membership function is given by

$$\mu(x) = 2(1 + \exp(\frac{\pi|x - et|}{\sqrt{6}\sigma t}))^{-1}, x \in \Re; \tag{2.2}$$

where $e$ and $\sigma$ are the drift and diffusion coefficients, respectively.

**Definition 2.9** (Liu, 2006) Suppose $\{\Theta, \mathcal{P}, \mathrm{Cr}\}$ is a credibility space and $\{\Omega, \mathcal{A}, \mathrm{Pr}\}$ is a probability space. The product $\{\Theta, \mathcal{P}, \mathrm{Cr}\} \times \{\Omega, \mathcal{A}, \mathrm{Pr}\}$ is called a chance space.

**Definition 2.10** (Li and Liu, 2008) Suppose $\{\Theta, \mathcal{P}, \mathrm{Cr}\} \times \{\Omega, \mathcal{A}, \mathrm{Pr}\}$ is a chance space. A chance measure of an event $\Lambda$ is defined as

$$Ch\{\Lambda\} = \begin{cases} \sup\limits_{\theta \in \Theta}(Cr\{\theta\} \wedge Pr\{\Lambda(\theta)\}), & \text{if } \sup\limits_{\theta \in \Theta}(Cr\{\theta\} \wedge Pr\{\Lambda(\theta)\}) < 0.5 \\ 1 - \sup\limits_{\theta \in \Theta}(Cr\{\theta\} \wedge Pr\{\Lambda^c(\theta)\}), & \text{if } \sup\limits_{\theta \in \Theta}(Cr\{\theta\} \wedge Pr\{\Lambda(\theta)\}) \geq 0.5 \end{cases} \tag{2.3}$$

**Definition 2.11** (Liu, 2006) A hybrid variable is defined as a measurable function from a chance space $\{\Theta, \mathcal{P}, \mathrm{Cr}\} \times \{\Omega, \mathcal{A}, \mathrm{Pr}\}$ to the set of real numbers such that for any Borel set B of real numbers the set

$$\{\xi \in B\} = \{(\theta, \omega) \in \Theta \times \Omega \mid \xi(\theta, \omega) \in B\} \tag{2.4}$$

is an event.

**Definition 2.12** (Liu, 2008) Suppose $T$ is an index set and $\{\Theta, \mathcal{P}, \mathrm{Cr}\} \times \{\Omega, \mathcal{A}, \mathrm{Pr}\}$ is a chance space. A hybrid process is defined as a measurable function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr}\} \times \{\Omega, \mathcal{A}, \mathrm{Pr}\}$ to the set of real numbers such that for each $t \in T$ and any Borel set B of the real numbers, the set $\{(\theta, \omega) \in \Theta \times \Omega \mid X(t, \theta, \omega) \in B\}$ is an event.

**Definition 2.13** (Liu, 2008) Suppose $B_t$ is a standard Brownian motion and $C_t$ is a standard Liu process. The process $D_t = (B_t + C_t)$ is called the D process or the Wiener-Liu process. Subsequently, the D process is said to be standard if both $B_t$ and $C_t$ are standard.

**Definition 2.14** (Liu, 2008) Suppose $\xi_1, \xi_2, \ldots$ are independent and identically distributed positive hybrid inter-arrival times. By definition, $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n \geq 1$. Then the hybrid process

$$N_t = \max_{n \geq 0} \{n \mid S_n \leq t\} \tag{2.5}$$

is called a hybrid renewal process.

**Definition 2.15** (Haugh, 2010) A filtration, $\{\mathcal{F}_t\}_{t \in [0,T]}$, models the flow of information over a specific period of time. Given a chance space $\{\Theta, \mathcal{P}, \mathrm{Cr}\} \times \{\Omega, \mathcal{A}, \mathrm{Pr}\}$, a filtration, $\{\mathcal{F}_t\}_{t \in [0,T]}$, is an increasing family of $\sigma$-algebras on $\Theta \times \Omega$, such that,

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \tag{2.6}$$

for $s \leq t$.

**Definition 2.16** Suppose $V_t$ is a hybrid process and $P_t$ is a certain given level. Then the hybrid variable

$$\tau_{P_t} = \inf\{t \geq 0 \mid V_t = P_t\} \tag{2.7}$$

is the first hitting time that $V_t$ reaches the level $P_t$.

Randomness and fuzziness are two basic forms of indeterminacy. In order to model these forms of indeterminacy, mathematical theories have been developed. Probability theory (Kolmogorov, 1933) models randomness and fuzzy set theory (Zadeh, 1965) deals with fuzziness. However, in some cases, randomness and fuzziness appear simultaneously in a mathematical system. This gave Li and Liu (2008) the impetus to pioneer the notion of a chance measure in order to measure hybrid events. Chance theory, which is a counterpart of probability theory and fuzzy theory, was developed in order to deal with the dynamics of phenomena with both fuzziness and randomness.

Various scholars have examined the mechanics of CPPI strategies using probability theory (such as, Neftci, 2008 and Cont and Tankov, 2007) and fuzzy set theory (for example, Matenda, 2016). The main goal of this study is to analyse the mechanics of CPPI strategies in hybrid markets using chance theory. Hybrid markets are markets where asset prices are assumed to be driven by hybrid processes which model both randomness and fuzziness. A participation rate is influenced by a myriad of factors. The relationship between the CPPI-insured portfolio value and the multiplier value is examined. Moreover, the correlation between the multiplier value and the risk of the CPPI-insured portfolio is analysed.

Conceptually, CPPI strategies guarantees the initial principal investment at the end of the investment horizon. However, in reality, CPPI techniques are exposed to gap risk which emanates from sudden significant downward asset price jumps. In practice, asset prices do jump in response to unexpected events and news such as wars, market crashes, acts of terrorism and civil unrest. Gap risk is the possibility that the value of the CPPI-insured portfolio may crash below the lower bound known as the floor. A CPPI strategy exhibits a loss when $P_t \geq V_t$. This peace of research work constructs a strong foundation for the calculation of the multiplier value in accordance to the risk tolerance of the market participants. The study opens up a way to analytically determine gap risk measures for CPPI approaches in hybrid markets. This research paper is the first peace of work to analyse CPPI strategies in hybrid markets.

## 3. CPPI Strategies in Hybrid Markets

A financial market of interest is assumed to contain two assets: an underlying risky asset (usually stock) and a default-free security (such as a riskless government bond).

**CPPI Strategies in the Absence of Jumps in Asset Prices**

Asset prices are considered to follow a basic hybrid Liu stock model with constant interest rate $r$ and constant volatility $\sigma$. The price process of a risky asset, $S_t$, at time $t$ follows a hybrid differential equation given below

$$\frac{dS_t}{S_t} = \mu dt + \sigma_1 dB_t + \sigma_2 dC_t, \tag{3.1}$$

where $\mu$ is the stock drift, $\sigma_1$ is the random stock diffusion, $\sigma_2$ is the fuzzy stock diffusion, $B_t$ is a standard Brownian motion and $C_t$ is a standard Liu process. Subsequently, the exact solution of equation (3.1) is

$$S_t = e^{(\mu - \frac{\sigma_1^2}{2})t + \sigma_1 B_t + \sigma_2 C_t}. \tag{3.2}$$

The price process of a risk-free bond, $B_t$, at time $t$ evolves according to the following hybrid differential equation

$$\frac{dB_t}{B_t} = rdt, \tag{3.3}$$

where $r$ is the riskless interest rate. Explicitly, the solution of equation (3.3) is

$$B_t = B_0 e^{rt}. \tag{3.4}$$

The basic assumption here is that $\mu > r$.

Contemplating the mechanics of a CPPI technique, the cushion is described by a hybrid differential equation given by

$$\frac{dU_t}{U_t} = (m(\mu - r) + r)dt + m\sigma_1 dB_t + m\sigma_2 dC_t, \tag{3.5}$$

whose explicit solution is

$$U_t = e^{((m(\mu - r) + r) - \frac{m^2\sigma_1^2}{2})t + m\sigma_1 B_t + m\sigma_2 C_t}. \tag{3.6}$$

Convincingly, the conclusion that can be deduced by pondering upon the mechanics of the cushion is that in a basic hybrid Liu stock model with continuous trading, CPPI techniques are not exposed to gap risk. When financial markets are falling the cushion also falls and in a worst case scenario the cushion becomes zero. That is, the cushion cannot be negative.

The value of the CPPI-insured portfolio is given by

$$V_t = P_t + e^{((m(\mu - r) + r) - \frac{m^2\sigma_1^2}{2})t + m\sigma_1 B_t + m\sigma_2 C_t}. \tag{3.7}$$

Equation (3.7) propounds that in a hybrid Liu stock model with continuous trading, CPPI strategies always work whatever the participation rate.

Furthermore, the expected value of the CPPI-insured portfolio is described by

$$E[V_t] = P_t + e^{((m(\mu - r) + r) - \frac{m^2\sigma_1^2}{2})t}. \tag{3.8}$$

Formula (3.8) indicates that, if $\mu > r$, the expected return of the CPPI-insured portfolio can be continuously increased by taking higher and higher multiplier values without assuming any additional underlying risk.

Conclusively, CPPI techniques always work. In a basic hybrid Liu stock model with continuous trading, CPPI strategies are not exposed to gap risk, regardless of the participation rate. A geometric Wiener-Liu process is a continuous path process. In continuous time diffusion models stock prices exhibit no jumps. The pay-off of an investor at time $T$ is described by

$$max\{V_T, B_T\} \equiv B_T + max\{U_T, 0\}. \tag{3.9}$$

However, in practice, asset prices do jump in response to unexpected events and news such as wars and acts of terrorism. Empirical evidence indicates that CPPI techniques are exposed to gap risk which originates from sudden significant downward asset price shocks. Gap risk is magnified by time constraints encountered in rebalancing the portfolio before it crashes down below the floor. Illiquid market for the underlying risky asset promotes jumps in asset market prices.

## Gap Risk for CPPI Strategies in Hybrid Markets

From $t = 0$ to time $\tau$ where $V_t$ touches $P_t$ the portfolio value evolves according to the following hybrid differential equation

$$dV_t = (U_t + P_t - mU_t)\frac{dB_t}{B_t} + mU_t\frac{dS_t}{S_t}, \tag{3.10}$$

which can be inscribed as

$$\frac{dU_t}{U_t} = \frac{dB_t}{B_t} - m\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}, \tag{3.11}$$

$$\frac{dU_t}{U_t} = (1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}. \tag{3.12}$$

Assuming that $dR_t$ is a hybrid process which describes the relative change in the cushion, and is given by

$$dR_t = (1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}. \tag{3.13}$$

By substitution, equation (3.12) reduces to

$$\frac{dU_t}{U_t} = dR_t. \tag{3.14}$$

Applying the notion of a discounted cushion, $U_t^* = \frac{U_t}{B_t}$, adopted from Cont and Tankov (2007), equation (3.14) becomes

$$U_t^* = U_0^* e^{R_t}. \tag{3.15}$$

Subsequently, the pay-off of an investor at $t = 0$ can be described by

$$1 + max\{U_T^*, 0\} \equiv 1 + max\{\frac{U_T}{B_T}, 0\}. \tag{3.16}$$

After time $\tau$ the discounted cushion remains unchanged and the value of the discounted CPPI-insured portfolio is given by

$$\frac{V_t}{B_t} = 1 + (\frac{V_0}{B_0} - 1)e^{R_{t\wedge\tau}} \tag{3.17}$$

Equation (3.17) indicates that the exponential term can become negative when asset prices do jump. Therefore, CPPI-insured portfolios are exposed to gap risk. A CPPI-insured portfolio manifests a loss when $V_t \le P_t$ or alternatively, $U_t \le 0$ or equivalently, $U_t^* \le 0$, for $\in [0, T]$ (Matenda, 2016). The events, $\{U_t^* \le 0\}$, $\{U_t \le 0\}$ and $\{\tau \le T\}$ show that CPPI-insured portfolios register losses during the investment period [0, T].

## CPPI Strategies in the Presence of Jumps in Asset Prices

In practice, stock prices are not continuous but they do jump in response to unexpected events and news. In order to analyse the mechanics of CPPI strategies in the presence of jumps in asset prices, the study adopts a hybrid stock model with jumps proposed by Liu (2008).

The price process of the underlying risky asset, $S_t$, at time $t$ is assumed to evolve according to the following hybrid differential equation with jumps

$$\frac{dS_t}{S_t} = \mu dt + \sigma_1 dB_t + \sigma_2 dC_t + \lambda dN_t, \tag{3.18}$$

where $\mu$ is the stock drift, $\sigma_1$ is the random stock diffusion, $\sigma_2$ is the fuzzy stock diffusion, $B_t$ is a standard Brownian motion, $C_t$ is a standard Liu process, $\lambda$ is the stock renewal coefficient and $N_t$ is a hybrid renewal process.

Explicitly, the solution of equation (3.18) is given by

$$S_t = e^{(\mu - \frac{\sigma_1^2}{2})t + \sigma_1 B_t + \sigma_2 C_t + \lambda N_t} \tag{3.19}$$

The price process of a default-free bond, $B_t$, at time $t$ is described by the following hybrid differential equation

$$dB_t = B_t r dt, \tag{3.20}$$

where $r$ is the risk-free interest rate. Exactly, the solution of equation (3.20) is

$$B_t = B_0 e^{rt}. \tag{3.21}$$

It is asumed that $\mu > r$.

In this case, the cushion satisfies a hybrid differential equation given by

$$\frac{dU_t}{U_t} = (m(\mu - r) + r)dt + m\sigma_1 dB_t + m\sigma_2 dC_t + m\lambda dN_t, \tag{3.22}$$

whose explicit solution is

$$U_t = e^{((m(\mu - r) + r) - \frac{m^2\sigma_1^2}{2})t + m\sigma_1 B_t + m\sigma_2 C_t + m\lambda N_t}. \tag{3.23}$$

From equation (3.22), $dU_t$ can become negative. As a result, the initial principal investment is no longer guaranteed at the end of the investment period.

The CPPI-insured portfolio value in this case is described by

$$V_t = P_t + e^{((m(\mu-r)+r)-\frac{m^2\sigma_1^2}{2})t+m\sigma_1 B_t+m\sigma_2 C_t+m\lambda N_t}. \tag{3.24}$$

Subsequently, the expected value of the CPPI-insured portfolio is given by

$$E[V_t] = P_t + e^{((m(\mu-r)+r)-\frac{m^2\sigma_1^2}{2})t+m\lambda N_t}. \tag{3.25}$$

Practically, it is not feasible to continuously increase the CPPI-insured portfolio return by taking higher and higher participation rates without taking additional risk. The multiplier value which is greater than one ($m > 1$) magnifies shocks in the CPPI-insured portfolio value. In bullish markets, the higher the value of the multiplier, the higher the speed at which the insured portfolio value increases. However, in bearish markets, the higher the value of the multiplier, the higher the speed at which the insured portfolio value falls towards the floor. The risk of a loss in a CPPI strategy is positively correlated with the multiplier value.

The multiplier value is influenced by a multiplicity of factors. This peace of research work constructs a strong foundation for the calculation of the multiplier value in accordance with the risk tolerance of the investors. The study opens up a way to analytically determine gap risk for CPPI approaches in hybrid markets. Gap risk for CPPI techniques has to be calculated. The participation rate should be determined by relating it to specific gap risk measures (Cont and Tankov, 2007).

A basic hybrid Liu stock model is a continuous-path model. Diffusion models are not good enough to model asset prices because asset prices do jump. Moreover, continuous-path models are based on the use of a Gaussian distribution which undervalues the possibility of extreme events (Matenda, 2016).

Matenda (2016) proposes that jump-diffusion models, such as a hybrid stock model with jumps, are realistic asset price models because various risk types cannot be modelled by diffusion models, risk-neutral returns are regarded as non-Gaussian and leptokurtic, and financial markets are incomplete. In incomplete markets asset prices exhibit jumps.

## 4. Conclusions

In several mathematical systems, randomness and fuzziness appear simultaneously. In order to model the dynamic phenomena with both randomness and fuzziness, chance theory has been developed. This study has analysed the mechanics of CPPI strategies in hybrid markets with both randomness and fuzziness. The research paper is the first peace of work to apply chance theory to CPPI strategies.

Assuming a basic hybrid Liu stock model with continuous trading, CPPI techniques are not exposed to gap risk regardless of the participation rate. In a hybrid Liu stock model environment, assuming that $\mu > r$, the expected return of the CPPI-insured portfolio can be continuously increased, without taking additional risk, by taking higher and higher multiplier values. However, empirical evidence indicates that CPPI techniques are exposed to gap risk which originates from sudden significant downward asset price jumps. The multiplier value which is greater than one ($m > 1$) magnifies shocks in the insured portfolio value. It is substantiated that there is a direct relationship between $m$ and $V_t$. The insured portfolio volatility is positively correlated with the participation rate. This research paper develops a strong foundation for the determination of $m$ based on the risk tolerance of the investors. Gap risk for CPPI strategies has to be quantified. The study opens a way for the analytical quantification of gap risk for CPPI techniques. The participation rate has to be gazetted by relating it to specific gap risk measures. Jump-diffusion models are realistic models for asset prices because they allow security prices to jump whilst maintaining the independence and stationarity of their returns. Matenda (2016) propounds that the mathematical tractability of jump-diffusion models makes it possible to execute financial calculations and present their complicated results in an easy way. As a further development of this paper, in the future the author seeks to analytically quantify gap risk for CPPI strategies in hybrid financial markets.

## References

[1]. Black, F., and Jones, R. C.: *Simplifying Portfolio Insurance*, The Journal of Portfolio Management, vol. 14, no. 1, pp. 48-51 (1987).

[2]. Black, F., and Scholes, M.: *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, vol. 81, pp. 637-654 (1973).

[3]. Cont, R., and Tankov, P.: *Constant Proportion Portfolio Insurance in the Presence of Jumps in Asset Prices*, Columbia University Center of Financial Engineering, Financial Engineering Report No. 2007-10 (2007). [Online] Available at: http://ssrn.com/abstract=1021084

[4]. Gao, J.: *Credibilistic Option Pricing: A New Model*, Journal of Uncertain Systems, vol. 2, no. 4, pp. 3-7 (2008).

[5]. Haugh, M.: *Introduction to Stochastic Calculus*, Lecture Notes in Financial Engineering: Continuous-Time Models (2010).

[6]. Ito, K.: *Stochastic Integral*, Proceedings of the Japan Academy Series A, vol. 20, no. 8, pp. 519-524, (1944).

[7]. Jiwo, S., and Chikodza, E.: *A Hybrid Optimal Control Model*, Journal of Uncertain Systems, vol. 9, no. 1, pp.

3-9 (2015).

[8]. Kallenberg, O.: *Foundations of Modern Probability* (*Second Edition*), Springer, New York (1997).

[9]. Kolmogorov, A. N.: *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Julius Springer, Berlin (1933).

[10]. Kunita, H., and Watanabe, S.: *On Square Integrable Martingales*, Nagoya Math, 30, 209-245 (1967).

[11]. Kwakernaak, H.: *Fuzzy Random Variables-II: Algorithms and Examples for the Discrete Case*, Information Sciences, vol. 17, pp. 253-278 (1979).

[12]. Kwakernaak, H.: *Fuzzy Random Variables-I: Definitions and Theorems*, Information Sciences, vol. 15, pp. 1-29 (1978).

[13]. Leland, H. E.: *Who Should buy Portfolio Insurance?*, The Journal of Finance, vol. 35, no. 2, pp. 581-594 (1980).

[14]. Li, X., and Liu, B.: *Chance Measure for Hybrid Events with Fuzziness and Randomness*, Soft Computing, vol. 13, no. 2, pp. 105-115 (2008).

[15]. Li, X., and Liu, B.: *A Sufficient and Necessary Condition for Credibility Measures*, International Journal of Uncertainty, Fuzziness & Knowledge-Based Systems, vol. 14, no. 4, pp. 527-535 (2006).

[16]. Liu, B.: *Uncertain Theory (Fifth Edition)*, Uncertainty Theory Laboratory, Beijing (2015). [Online] Available at: http://orsc.edu.cn/liu/ut.pdf

[17]. Liu, B.: *Uncertainty Theory (Fourth Edition)*, Uncertainty Theory Laboratory, Beijing (2012). [Online] Available at: http://orsc.edu.en/liu/ut.pdf

[18]. Liu, B.: *Fuzzy Process, Hybrid Process and Uncertain Process*, Journal of Uncertain Systems, vol. 2, no. 1, pp. 3-16 (2008).

[19]. Liu, B.: *Uncertainty Theory (Second Edition)*, Springer-Verlag, Berlin (2007).

[20]. Liu, B.: *A Survey of Credibility Theory*, Fuzzy Optimisation and Decision Making, vol. 5, no. 4, pp. 387-408 (2006).

[21]. Liu, B.: *Uncertainty Theory*, Springer-Verlag, Berlin (2004). [Online] Available at: www.jus.org.uk

[22]. Liu, B.: *Theory and Practice of Uncertain Programming*, Physica-Verlag, Heidelberg (2002).

[23]. Liu, B., and Liu, Y. K.: *Expected Value of Fuzzy Variable and Fuzzy Expected Value Models*, IEEE Transactions on Fuzzy Systems, vol. 10, no. 4, pp. 445-450 (2002).

[24]. Matenda, F. R.: *Constant Proportion Portfolio Insurance Strategies in Fuzzy Financial Markets*, Journal of Mathematics and Statistical Science, vol. 2016, pp. 48-63 (2016).

[25]. Matenda, F. R., Chikodza, E., and Gumbo, V.: *Measuring Gap Risk for Constant Proportion Portfolio Insurance Strategies in Uncertain Markets*, Journal of Mathematics and Statistical Science, vol. 2015, pp. 18-31 (2015).

[26]. Merton, R.: *Theory of Rational Option Pricing*, Bell Journal of Economics and Management Science, vol. 4, no. 1, pp. 141-183 (1973).

[27]. Neftci, S. N.: *Principles of Financial Engineering (Second Edition)*, Elsevier Inc, London (2008).

[28]. Peng, J.: *Risk Metrics of Loss Function for Uncertain System*, Fuzzy Optimization and Decision Making, vol. 12, no. 1, pp. 53-64 (2013).

[29]. Peng, J.: *A General Stock Model For Fuzzy Markets*, Journal of Uncertain Systems, vol. 2, no. 4, pp. 248-254 (2008). [Online] Available at: www.jus.org.uk

[30]. Perold, A. F.: *Constant Proportion Portfolio Insurance*, Harvard Business School, (1986).

[31]. Qin, Z., and Li, X.: *Fuzzy Calculus for Finance*, Uncertainty Theory Laboratory, Beijing (2008). [Online] Available at: http://orsc.edu.cn/process/fc.pdf

[32]. Samuelson, P.: *Proof that Properly Anticipated Prices Fluctuate Randomly*, Industrial Management Review, vol. 6, pp. 41-50 (1965).

[33]. Schied, A.: *Model-free CPPI*, Journal of Economic Dynamics and Control, vol. 40, pp. 84-94 (2014).

[34]. Trippi, R. R., and Harriff, R. B.: *Dynamic Asset Allocation Rules: Survey and Synthesis*, The Journal of Portfolio Management, vol. 17, no. 4, pp. 19-26 (1991).

[35]. Yoshida, Y.: *A Discrete-time Model of American Put Option in an Uncertain Environment*, European Journal of Operational Research, vol. 151, no. 1, pp. 153-166 (2003).

[36]. Yoshida, Y., Yasuda, M., Nakagami, J., and Kurano, M.: *A New Evaluation of Mean Value for Fuzzy Numbers and its Application to American Put Option Under Uncertainty*, Fuzzy Sets and Systems, vol. 157, no. 19, pp. 2614-2626 (2006).

[37]. You, C.: *Existence and Uniqueness Theorems for Fuzzy Differential Equations*, (2008). [Online] Available at: http://orsc.edu.cn/process/080316.pdf.

[38]. Zadeh, L. A.: *Fuzzy Sets*, Information and Control, vol. 8, pp. 338-353 (1965).

[39]. Zhao, R., and Liu, B.: *Renewal Process with Fuzzy Interarrival Times and Rewards*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 11, no. 5, pp. 573-586 (2003).