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Comparison of Estimation Methods for the Parameters of Poisson-Lomax Distribution under Progressive Type-II Censoring

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Abstract

The Poisson-Lomax distribution, with decreasing and upside down shapes of failure rate, is considered as a lifetime distribution. Its genesis may appear in the complementary risks model and parallel systems. Based on progressive type-II censoring, the maximum likelihood, unweighted least squares, weighted least squares and Bayes (using linear-exponential and general entropy loss functions) estimation methods are considered to estimate the involved parameters. The performance of these methods is compared through an extensive numerical simulation, based on mean squared errors and relative absolute biases of the estimates. Two real data sets are used to compare the Poisson-Lomax distribution with the exponentiated Lomax distribution, exponentiated Weibull Poisson distribution, exponentiated exponential geometric distribution, exponentiated exponential Poisson distribution and Lomax distribution which have showed that the former distribution is better to fit the data than the other five distributions.

Keywords: Complementary risks model; Poisson-Lomax distribution; Progressive type-II censoring; Maximum likelihood estimation; Unweighted and weighted least squares estimations; Bayes estimation; Simulation.

1. Introduction

Pareto (1897) proposed his distribution as a model for the distribution of income. There are different forms of the Pareto distribution in statistical literature, which were used as models in the fields of insurance, business, economics, engineering, hydrology, reliability and other areas as well, see for example Arnold (1983), Johnson et al. (1994), Ali Mousa (2003), Nigm et al. (2003) and Raqab et al. (2010).

Lomax (1954) suggested the use of Pareto distribution of the second kind known as Lomax distribution (LD) as a model for business failure data. Bain and Engelhardt (1992) proposed LD as a model for biomedical problems, such as survival time following a heart transplant. Howlader and Hossain (2002) proposed Bayesian survival estimation of Pareto distribution of the second kind based on failure-censored data. Soliman (2008) discussed Bayesian and maximum likelihood estimates (MLEs) for the parameters, reliability, and hazard functions based on a general progressively type-II censored data from LD. Cramer and Schmiedt (2011) studied competing risks model based on LD under progressive type-II censoring.

In reliability and survival analysis, engineering, demographic, actuarial literature, econometrics, biological or medical studies, units might fail owing to one of several risk factors. Basu and Klein (1982) established an idea of the complementary risks (CR) model which describes the lifetime of a parallel system. In CR model, if the risks are latent, then a difficulty arises about which factor was responsible for the component failure and hence the lifetime associated with a particular risk can not be observed. It can be observed only, in this case, the maximum lifetime value among all risks.

In the statistical literature, several distributions have been introduced to model lifetime data by compounding some distributions. Adamidis and Loukas (1998) and Kuş (2007) introduced the exponential geometric and exponential Poisson distributions, respectively, which have decreasing failure rate, and studied their properties, while Barreto-Souza and Cribari-Neto (2009) added a power parameter to the distribution proposed by Kuş (2007). Cancho et al. (2011) and Louzada et al. (2011) obtained thePoisson exponential (PE) and the complementary exponential geometric distributions, which have increasing failure rate, and studied their properties. Louzada et al. (2013) proposed the complementary exponentiated exponential geometric distribution, which has a one shape and two scale parameters accommodating increasing, decreasing and bathtub failure rates and discussed their properties. Based on the CR model, Mahmoudi and Sepahdar (2013) proposed a four-parameter distribution, which has an increasing, decreasing, bathtub-shaped and unimodal failure rates known as the exponentiated Weibull Poisson distribution (EWPD). Tomazella et al. (2013) considered a Bayesian reference analysis for the PE distribution following the technique presented in Cancho et al. (2011). Singh et al. (2014) considered the estimation problem of the parameters of PE distribution using maximum likelihood (ML) and Bayes procedures. Rezaeia et al. (2013) and Ristić and Nadarajah (2014) proposed the exponentiated exponential geometric distribution (EEGD) and exponentiated exponential Poisson distribution (EEPD), respectively, which have decreasing, increasing and upside-down bathtub failure rates. AL-Zahrani and Sagor (2014) considered the Poisson-Lomax distribution (PLD) and studied its properties.

In medical or industrial applications, censoring usually applies when the experimenter is unable to get total information on lifetimes for each unit or reducing the total test time and the associated cost. Type-I and type-II are two commonly used censoring schemes (CSs), see for example, Mann et al. (1974), Meeker and Escobar (1998) and Lawless (2003). These types of censoring cannot allow the experimenter to remove units from a life test at various stages during the experiment. The experimenter can overcome this problem by using progressive type-II censoring which is considered to be a generalization of type-II censoring. It allows the experimenter to remove units from a life test at various stages during the experiment, see Balakrishnan and Aggarwala (2000).

In this paper, we consider the PLD, with decreasing and upside down shapes of failure rate, as a lifetime distribution under CR model. Five estimation methods for the parameters, based on progressive type-II censoring, are discussed and compared through a simulation study. we compare among PLD, exponentiated Lomax distribution (ELD), EEPD, EEGD, EWPD and LD based on two real data sets.

The rest of the paper is organized as follows: Section 2, presents the PLD. Some estimation methods are discussed in Section 3. Applications of PLD to two real data sets are given in Section 4. Simulation study followed by conclusions are presented in Sections 5 and 6, respectively.

2. Formulation of PLD under CR Model

Following AL-Zahrani and Sagor (2014), the PLD can be derived as follows: Assume that K is a random variable, with realization k, denoting the number of CR associated with the occurrence of a given event. If K has a zero truncated Poisson distribution, then its probability mass function (PMF) is given by

$$P(K=k)=\frac{\alpha^{k}e^{-\alpha}}{k!(1-e^{-\alpha})}, \qquad k=1,2,\ldots,(\alpha>0). \tag{2.1}$$

Suppose that $X_1,X_2,\ldots,X_k$ denote the failure times due to k CR and X_i, $i=1,\ldots,k$ has LD with probability density function (PDF) and cumulative distribution function (CDF) given, respectively, by

$$f_X(x;\beta,\gamma)=\frac{\gamma}{\beta}(1+\frac{x}{\beta})^{-(\gamma+1)},\qquad x>0,\quad (\beta,\gamma>0), \tag{2.2}$$

$$F_X(x;\beta,\gamma)=1-\left(1+\frac{x}{\beta}\right)^{-\gamma}, \tag{2.3}$$

where β and γ are scale and shape parameters, respectively.

Since only the largest of $X_1, X_2, \ldots, X_k$ is usually observed for CR, then we write

$$Z=\max\{X_1,X_2,\ldots,X_k\},$$

to denote the overall failure time of a test unit. Then the PDF and CDF of Z can be derived as follows:

The conditional density function of Z, given $K=k$, is given by

$$f(z\mid k)=k[F_X(z)]^{k-1}f_X(z)=\frac{k\gamma}{\beta}\left[1-\left(1+\frac{z}{\beta}\right)^{-\gamma}\right]^{k-1}\left(1+\frac{z}{\beta}\right)^{-(\gamma+1)}.$$

The marginal PDF of Z is given by the countable mixture

$$\begin{aligned} f(z)&=\sum_{k=1}^{\infty}f(z\mid k)P(K=k)\\ &=\frac{\gamma\alpha}{\beta}\left(1+\frac{z}{\beta}\right)^{-(\gamma+1)}\frac{e^{-\alpha}}{1-e^{-\alpha}}\sum_{k=1}^{\infty}\frac{\left[\alpha(1-\left(1+\frac{z}{\beta}\right)^{-\gamma})\right]^{k-1}}{(k-1)!}\\ &=\frac{\gamma\alpha}{\beta}\left(1+\frac{z}{\beta}\right)^{-(\gamma+1)}\frac{e^{-\alpha\left(1+\frac{z}{\beta}\right)^{-\gamma}}}{1-e^{-\alpha}}. \end{aligned} \tag{2.4}$$

Therefore, the CDF of Z is given by

$$F(z)=\int_0^z f(y)\,dy=\frac{e^{-\alpha(1+\frac{z}{\beta})^{-\gamma}}-e^{-\alpha}}{1-e^{-\alpha}} \tag{2.5}$$

The survival function (SF) and hazard rate function (HRF) of PLD with CDF (2.5) are given, respectively, by

$$S(z)=\frac{1-e^{-\alpha(1+\frac{z}{\beta})^{-\gamma}}}{1-e^{-\alpha}}, \tag{2.6}$$

$$h(z)=\frac{\gamma\alpha}{\beta}\frac{\left(1+\frac{z}{\beta}\right)^{-(\gamma+1)}}{e^{\alpha(1+\frac{z}{\beta})^{-\gamma}}-1}. \tag{2.7}$$

Remark 1. If $\alpha\to 0$, then CDF (2.5) of PLD is reduced to traditional CDF (2.3) of LD.

PDF (2.4) and HRF (2.7) of PLD are plotted in Figure 1 for different values of α,β and γ. It can be noticed from this figure that the PDFs and HRFs are decreasing and unimodal. Table 1 displays the mean, median, mode and variance of PLD for different values of α,β and γ.

From Table 1, it can be noticed that:

1. For fixed values of β and γ, by increasing α, the mean, median, mode and variance increase.
2. For fixed values of α and γ, by increasing β, the mean, median, mode and variance increase.
3. For fixed values of α and β, by increasing γ, the mean, median, mode and variance decrease.

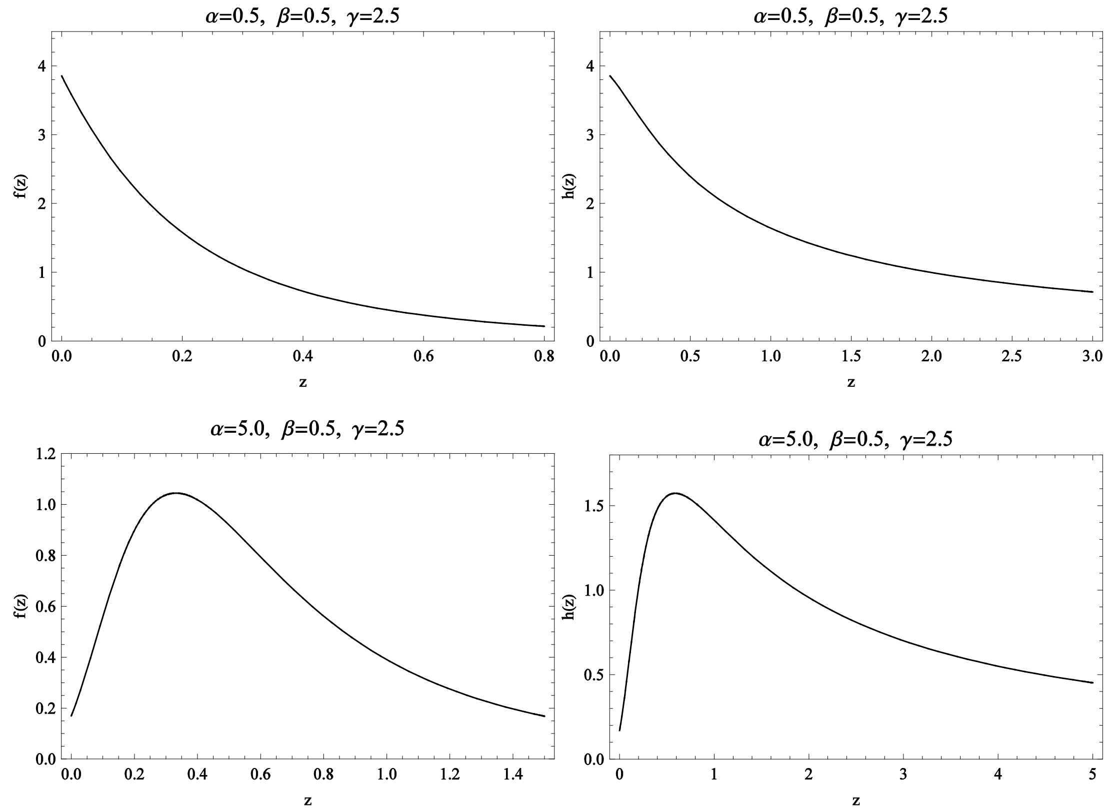

Figure 1. Left (Right) panel: PDF (HRF) of PLD for different values of α, β and γ.

Table 1. The mean, median, mode and variance of PLD for different values of α, β and γ.

		$\beta = 0.5$			$\beta = 2.0$		
		$\alpha = 1.0$	$\alpha = 5.0$	$\alpha = 10.0$	$\alpha = 1.0$	$\alpha = 5.0$	$\alpha = 10.0$
$\gamma = 2.5$	mean	0.4456	0.9239	1.3704	1.7823	3.6959	5.4816
	median	0.2364	0.6064	0.9543	0.9456	2.4257	3.8172
	mode	0.0000	0.3320	0.5978	0.0000	1.3279	2.3912
	variance	0.8264	2.1584	3.7435	13.2230	34.5350	59.8956
$\gamma = 5.0$	mean	0.1617	0.3053	0.4226	0.6469	1.2213	1.6904
	median	0.1068	0.2438	0.3527	0.4272	0.9751	1.4109
	mode	0.0000	0.1652	0.2641	0.0000	0.6607	1.0563
	variance	0.0349	0.0634	0.0840	0.5585	1.0143	1.3439

3. Estimation Methods

In this section, based on progressive type-II censoring, we consider five estimation methods to estimate the parameters α, β and γ. The methods are ML, unweighted least squares (UWLS), weighted least squares (WLS) and Bayes (using linear-exponential (LINEX) and general entropy (GE) loss functions) estimations.

Progressive type-II censoring can be applied as follows: Suppose that $m\,(<\ n)$ and R_1, R_2, …, R_m are fixed before the experiment. R_1 surviving units are randomly removed from the test when the first failure time occurs and R_2 surviving units are randomly removed from the test when the second failure time occurs. The test continues in the same manner until the m-th failure at which all the remaining surviving units $R_m = n - m - \sum_{j=1}^{m-1} R_j$ are removed from the test, thereby terminating the life test. The data from progressively type-II censored samples are as follows: ($z_{1:m:n}$; R_1), …, ($z_{m:m:n}$; R_m) where $z_{1:m:n} < \ldots < z_{m:m:n}$ denote the m ordered observed failure times and R_1, …, R_m denote the number of units removed from the experiment at failure times $z_{1:m:n}$, …, $z_{m:m:n}$.

1. Maximum Likelihood Estimation

The likelihood function under progressive type-II censoring from the PLD with PDF (2.4) and CDF (2.5) is given by

$$L(\boldsymbol{\theta};\mathbf{z}) \propto \prod_{i=1}^{m} f(z_i)[1-F(z_i)]^{R_i}, \tag{3.1}$$

where $\mathbf{z} = (z_1,\ldots,z_m), z_i \equiv z_{i:m:n}, i = 1,\ldots,m$ and $\boldsymbol{\theta} = (\theta_1 = \alpha, \theta_2 = \beta, \theta_3 = \gamma)$.

Based on Equations (2.4) and (2.5), the log-likelihood function takes the form

$$\pounds = \log[L(\boldsymbol{\theta};\mathbf{z})] \propto m\log\left[\frac{\gamma\alpha}{\beta}\right] - (\gamma+1)\sum_{i=1}^{m}\log\left[1+\frac{z_i}{\beta}\right] - \alpha\sum_{i=1}^{m}\left(1+\frac{z_i}{\beta}\right)^{-\gamma}$$

$$-\left(m+\sum_{i=1}^{m} R_i\right)\log\left[1-e^{-\alpha}\right]+\sum_{i=1}^{m} R_i \log\left[1-e^{-\alpha\left(1+\frac{z_i}{\beta}\right)^{-\gamma}}\right]. \tag{3.2}$$

The MLEs $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$ and $\hat{\gamma}_{ML}$ of α, β and γ could be obtained by solving the likelihood equations, $\frac{\partial £}{\partial \theta_i}=0$, $i=1,2,3$, with respect to θ_i. These MLEs can not be obtained in closed forms and hence a numerical iteration method for the likelihood equations should be used.

The local Fisher information matrix, $\mathbf{I}$, for $(\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\gamma}_{ML})$ is the 3×3 symmetric matrix of negative second partial derivatives of £ with respect to α, β and γ, see Nelson (1990). So that $\mathbf{I}$ is given by

$$\mathbf{I}=-\left(\frac{\partial^2 \hat{£}}{\partial \theta_i \partial \theta_j}\right)_{3\times 3}, \qquad i,j=1,2,3,$$

where $\boldsymbol{\theta}=(\theta_1=\alpha, \theta_2=\beta, \theta_3=\gamma)$ and the caret ˆ indicates that the derivative is calculated at $(\hat{\alpha}_{ML}, \hat{\beta}_{ML}, \hat{\gamma}_{ML})$. The elements of the matrix $\mathbf{I}$ can be easily obtained.

The inverse of $\mathbf{I}$ is the local estimate $\mathbf{V}$ of the asymptotic variance-covariance matrix of $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$. That is

$$\mathbf{V}=\mathbf{I}^{-1}=\left(\operatorname{cov}\left(\theta_i, \theta_j\right)\right)_{3\times 3}, \qquad i,j=1,2,3. \tag{3.3}$$

Following the general asymptotic theory of MLEs, the sampling distribution of

$$\frac{\hat{\alpha}_{ML}-\alpha}{\sqrt{\operatorname{var}(\hat{\alpha}_{ML})}}, \quad \frac{\hat{\beta}_{ML}-\beta}{\sqrt{\operatorname{var}(\hat{\beta}_{ML})}} \quad \text{and} \quad \frac{\hat{\gamma}_{ML}-\gamma}{\sqrt{\operatorname{var}(\hat{\gamma}_{ML})}},$$

can be approximated by a standard normal distribution which is useful in constructing confidence intervals (CIs) for the unknown parameters.

A two-sided $(1-\tau)100\%$ normal approximation CIs for the parameters α, β and γ can then be constructed as

$$\hat{\alpha}_{ML} \pm z_{\tau/2}\sqrt{\operatorname{var}(\hat{\alpha}_{ML})}, \quad \hat{\beta}_{ML} \pm z_{\tau/2}\sqrt{\operatorname{var}(\hat{\beta}_{ML})} \quad \text{and} \quad \hat{\gamma}_{ML} \pm z_{\tau/2}\sqrt{\operatorname{var}(\hat{\gamma}_{ML})},$$

where $z_{\tau/2}$ is the value of a standard normal random variable leaving an area $\tau/2$ to the right and $\sqrt{\operatorname{var}(\hat{\alpha}_{ML})}$, $\sqrt{\operatorname{var}(\hat{\beta}_{ML})}$ and $\sqrt{\operatorname{var}(\hat{\gamma}_{ML})}$ can be obtained from (3.3).

2. Unweighted and Weighted Least Squares Estimations

The UWLS estimates $\hat{\alpha}_{UW}$, $\hat{\beta}_{UW}$ and $\hat{\gamma}_{UW}$ of α, β and γ can be obtained by minimizing the following quantity with respect to α, β and γ.

$$\Psi_1 \equiv \Psi_1(\boldsymbol{\theta};\mathbf{z}) = \sum_{i=1}^{m}\left(\log\left[-\log\left[1-\frac{\widehat{F}(z_i)+\widehat{F}(z_{i-1})}{2}\right]\right]-\log\left[-\log\left[1-F(z_i)\right]\right]\right)^2,$$

where $\widehat{F}(z_i)$ is the empirical CDF which can be written as, see Meeker and Escobar (1998),

$$\widehat{F}(z_i) = 1-\prod_{j=1}^{i}(1-\widehat{p}_j), \quad i=1,\ldots,m,$$

where

$$\widehat{p}_j = \frac{1}{n-\left[\sum_{k=2}^{j} R_{k-1}\right]-j+1}, \quad j=1,\ldots,m,$$

where $\sum_{k=2}^{j} R_{k-1}$ is equal zero if $k > j$.

The UWLS estimates $\hat{\alpha}_{UW}$, $\hat{\beta}_{UW}$ and $\hat{\gamma}_{UW}$ can be obtained by solving the equations $\frac{\partial \Psi_1}{\partial \theta_i} = 0$, $i = 1,2,3$ with respect to θ_i.

On the other hand the WLS estimates $\hat{\alpha}_{W}$, $\hat{\beta}_{W}$ and $\hat{\gamma}_{W}$ of α, β and γ can be obtained by minimizing the following quantity with respect to α, β and γ.

$$\Psi_2 \equiv \Psi_2(\boldsymbol{\theta};\mathbf{z}) = \sum_{i=1}^{m} W_i \left(\log\left[-\log\left[1 - \frac{\widehat{F}(z_i) + \widehat{F}(z_{i-1})}{2} \right]\right] - \log\left[-\log\left[1 - F(z_i) \right]\right] \right)^2,$$

where W_i is the weight factor which was proposed by Faucher and Tyson (1988). It may be approximated by:

$$W_i = 3.3\frac{\widehat{F}(z_i) + \widehat{F}(z_{i-1})}{2} - 27.5\left(1 - \left(1 - \frac{\widehat{F}(z_i) + \widehat{F}(z_{i-1})}{2} \right)^{0.025} \right), \quad i = 1,2,3.$$

The WLS estimates $\hat{\alpha}_W$, $\hat{\beta}_W$ and $\hat{\gamma}_W$ can be obtained by solving the equations $\frac{\partial \Psi_2}{\partial \theta_i} = 0$, $i = 1,2,3$ with respect to θ_i.

3. Bayes Estimation

Symmetric loss functions may be inappropriate in many real life situations, because they give overestimation or underestimation of the parameters. Overestimation of the parameters can lead to more severe or less severe consequences than underestimation,or vice versa. For example, when we estimate the average reliable working life of the components of a spaceship or an aircraft, overestimation is usually more serious than underestimation. Therefore, research has been directed towards asymmetric loss functions. A number of asymmetric loss functions is introduced for use, among these, the LINEX loss function and the GE loss function. The estimators of the parameters under the asymmetric loss function demonstrate their superiority over the estimators obtained under symmetric loss function, see for example Canfield (1970), Zellner (1986), Srivastava and Tanna (2001), Soliman et al. (2012) and Singh et al. (2014).

1. Bayes estimation under LINEX loss function

Varian (1975) suggested the use of LINEX loss function to be of the form

$$\mathcal{L}(\delta) \propto e^{\nu\delta} - \nu\delta - 1, \qquad \nu \neq 0,$$

where $\delta = \hat{\Theta}_{BL} - \Theta$ and $\hat{\Theta}_{BL}$ is the LINEX estimate of Θ.

The Bayes estimate of Θ, based on the LINEX loss function, is given by

$$\hat{\Theta}_{BL} = \frac{-1}{\nu} \log[E(e^{-\nu\Theta} \mid \mathbf{z})]. \tag{3.4}$$

Suppose that the prior belief of the experimenter is measured by a function $\pi(\alpha,\beta,\gamma)$, where α is independent of β and γ, so that the prior density function is given by

$$\pi(\alpha,\beta,\gamma) = \pi_1(\alpha)\pi_2(\beta,\gamma). \tag{3.5}$$

Suppose that $\pi_1(\alpha)$ is lognormal (μ_1,σ_1) with density function

$$\pi_1(\alpha) = \frac{1}{\sigma_1\alpha\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\log\alpha-\mu_1}{\sigma_1}\right)^2}, \quad \alpha>0, \quad (-\infty<\mu_1<\infty, \sigma_1>0). \tag{3.6}$$

Let

$$\pi_2(\beta,\gamma) = \pi_3(\beta \mid \gamma)\pi_4(\gamma), \tag{3.6}$$

where $\pi_3(\beta \mid \gamma)$ is lognormal (μ_2,γ) and $\pi_4(\gamma)$ is lognormal (μ_3,σ_2) with respective densities,

$$\pi_3(\beta \mid \gamma) = \frac{1}{\gamma\beta\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\log\beta-\mu_2}{\gamma}\right)^2}, \quad \beta>0, \quad (-\infty<\mu_2<\infty, \gamma>0), \tag{3.7}$$

$$\pi_4(\gamma) = \frac{1}{\sigma_2\gamma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{\log\gamma-\mu_3}{\sigma_2}\right)^2}, \quad \gamma>0, \quad (-\infty<\mu_3<\infty, \sigma_2>0). \tag{3.8}$$

Using Equations (3.6)-(3.8), the joint prior density function (3.5) is then given by

$$\pi(\alpha,\beta,\gamma) \propto \frac{1}{\alpha\beta\gamma^2} e^{-\Delta}, \tag{3.9}$$

where $\Delta = \frac{1}{2}\left[\left(\frac{\log\alpha - \mu_1}{\sigma_1}\right)^2 + \left(\frac{\log\beta - \mu_2}{\gamma}\right)^2 + \left(\frac{\log\gamma - \mu_3}{\sigma_2}\right)^2\right]$.

From (3.1) and (3.9) the joint posterior density function of α, β and γ is then given by

$$\pi^*(\alpha,\beta,\gamma \mid \mathbf{z}) = \eta^{-1}\,\phi\xi, \quad \alpha,\beta,\gamma > 0, \tag{3.10}$$

where

$$\left.\begin{aligned} \phi &= \prod_{j=1}^{m}\left(\frac{\left(\frac{1-e^{-\alpha(1+\frac{z_j}{\beta})^{-\gamma}}}{1-e^{-\alpha}}\right)^{R_j}}{\left(1+\frac{z_j}{\beta}\right)^{\gamma+1} e^{\alpha\left(1+\frac{z_j}{\beta}\right)^{-\gamma}}}\right), \\ \xi &= \frac{1}{\beta^2\gamma}\frac{e^{-\Delta}}{1-e^{-\alpha}}, \\ \eta &= \int_0^\infty\int_0^\infty\int_0^\infty \phi\xi \quad \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma. \end{aligned}\right\} \tag{3.11}$$

From (3.4) and (3.10) the LINEX estimates of α, β and γ are then given, respectively, by

$$\hat{\alpha}_{BL} = \frac{-1}{\nu}\log\left[\eta^{-1}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\nu\alpha}\,\phi\xi\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\right],$$

$$\hat{\beta}_{BL} = \frac{-1}{\nu}\log\left[\eta^{-1}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\nu\beta}\,\phi\xi\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\right],$$

$$\hat{\gamma}_{BL} = \frac{-1}{\nu}\log\left[\eta^{-1}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\nu\gamma}\,\phi\xi\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\right].$$

2. Bayes estimation under GE loss function

The GE loss function, proposed by Calabria and Pulcini (1994), is given by

$$\mathcal{L}(\hat{\Theta}_{BG},\Theta) \propto \left(\frac{\hat{\Theta}_{BG}}{\Theta}\right)^{\nu} - \nu\log\left[\frac{\hat{\Theta}_{BG}}{\Theta}\right] - 1, \qquad \nu \neq 0.$$

The Bayes estimate of Θ, based on the GE loss function, is given by

$$\hat{\Theta}_{BG} = \left[E(\Theta^{-\nu}) \right]^{\frac{-1}{\nu}}. \tag{3.12}$$

From (3.10) and (3.12) the Bayes estimates of α, β and γ, based on the GE loss function, is then given, respectively, by

$$\hat{\alpha}_{BG} = \left[\eta^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \alpha^{-\nu} \, \phi \xi \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \right]^{\frac{-1}{\nu}},$$

$$\hat{\beta}_{BG} = \left[\eta^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \beta^{-\nu} \, \phi \xi \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \right]^{\frac{-1}{\nu}},$$

$$\hat{\gamma}_{BG} = \left[\eta^{-1} \int_0^\infty \int_0^\infty \int_0^\infty \gamma^{-\nu} \, \phi \xi \, \mathrm{d}\alpha \, \mathrm{d}\beta \, \mathrm{d}\gamma \right]^{\frac{-1}{\nu}},$$

where ϕ, ξ and η are given by (3.11).

Remark 1. In developing the Bayes estimates, we have supposed the LINEX and GE loss functions, although some other loss functions also can be easily incorporated.

4. Applications of PLD to Two Real Data Sets

In this section, we compare among PLD, ELD, EEPD, EEGD, EWPD and LD based on two real data sets as follows:

- The first data set:
  The data set corresponds to the amount of annual rainfall (in inches) during February recorded at Los Angeles Civic Center from 1965 to 2006. The data set is 0.23, 1.51, 0.11, 0.49, 8.03, 2.58, 0.67, 0.13, 7.89, 0.14, 3.54, 3.71, 0.17, 8.91, 3.06, 12.75, 1.48, 0.70, 4.37, 0.00, 2.84, 6.10, 1.22, 1.72, 1.90, 3.12, 4.13, 7.96, 6.61, 3.21, 1.30, 4.94, 0.08, 13.68, 0.56, 5.54, 8.87, 0.29, 4.64, 4.89, 11.02, 2.37. Madi and Raqab (2007) and Raqab et al. (2010) used the amount of annual rainfall (in inches) recorded at Los Angeles Civic Center as real data.
- The second data set:
  We consider the data set consisting of thirty successive values of March precipitation (in inches) in Minneapolis/St Paul given in Hinkley (1977). The data set is 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05. This data set is also studied by Barreto-Souza and Cribari-Neto (2009).

For the first and second data sets, we compare the PLD with ELD, EEPD, EEGD, EWPD and LD through Kolmogorov-Smirnov (K-S) statistic, P-value, Akaike information criterion (AIC), consistent AIC (CAIC) and Bayesian information criterion (BIC), where

$$AIC = 2b - 2\pounds(\hat{\Omega}), \qquad CAIC = \frac{2bm}{m-b-1} - 2\pounds(\hat{\Omega}), \qquad BIC = b\log[m] - 2\pounds(\hat{\Omega}),$$

where $\hat{\Omega}$ is the MLE of Ω, $\pounds(\hat{\Omega})$ is the log-likelihood function calculated at $\hat{\Omega}$, b is the number of parameters and m is the sample size. The results are listed in Table 2 in which we can notice that the PLD fits the given two data sets better than the ELD, EEPD, EEGD, EWPD and LD. This is done graphically by plotting the empirical CDF against the CDF of PLD, ELD, EEPD, EEGD, EWPD and LD, see Figure 2.

The mean, median, mode and variance of PLD for MLEs of the parameters are presented in Table 3. The graphs of PDF and HRF of PLD for the first and second data sets are drawn in Figures 3 and 4.

Table 2. The MLEs, K-S statistic, P-value, AIC, CAIC and BIC.

The first data set									
Model	α	β	γ	δ	K-S	P-value	AIC	CAIC	BIC
PLD	0.752335	4.253500	2.256430	− − −−	0.100529	0.78968	204.741	205.373	209.954
ELD	− − −−	1.504270	0.891119	1.000690	0.127507	0.50194	212.563	213.195	217.776
EEPD	0.117117	0.185625	− − −−	0.098868	0.624513	1.42×10^{-14}	281.273	281.904	286.486
EEGD	0.472062	0.624496	− − −−	0.771739	0.443312	1.352×10^{-7}	271.888	272.520	277.101
EWPD	14.14710	1.456770	0.314837	0.281442	0.284897	0.00219	215.596	216.228	220.809
LD	− − −−	0.835917	0.687723	− − −−	0.151948	0.28672	215.498	215.806	218.973

The second data set									
Model	α	β	γ	δ	K-S	P-value	AIC	CAIC	BIC
PLD	5.62753	10.8902	16.44600	− − −−	0.058916	0.99994	82.9254	83.8485	87.1290
ELD	− − −−	0.31152	0.995958	3.78815	0.253782	0.04195	102.997	103.921	107.201
EEPD	5.24382	0.06407	− − −−	0.67426	0.328372	0.00310	117.518	118.441	121.722
EEGD	0.34749	0.78001	− − −−	1.90417	0.151056	0.50032	87.7545	88.6776	91.9581
EWPD	2.65188	1.28692	0.944423	1.91978	0.110572	0.85674	83.8742	84.7973	88.0778
LD	− − −−	0.83592	0.859654	− − −−	0.262857	0.03167	117.137	117.581	119.939

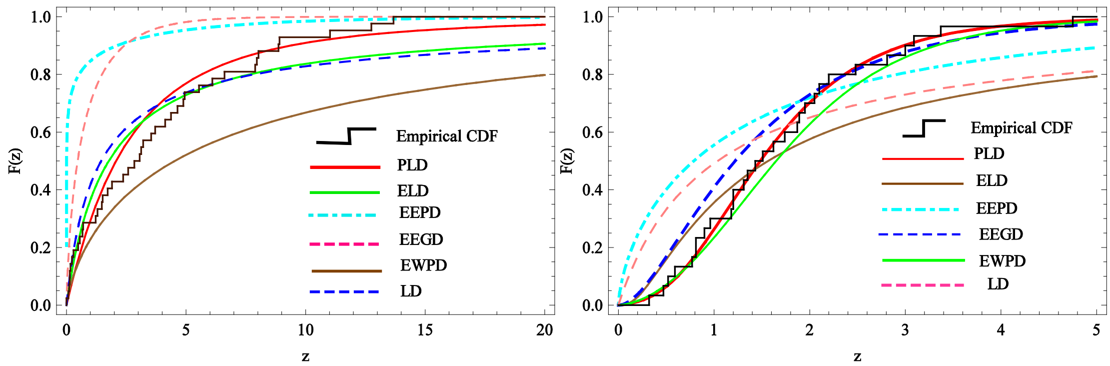

Figure 2. Left (Right) panel: Empirical CDF against CDF of PLD, ELD, EEPD, EEGD, EWPD and LD for the first (second) data set.

Table 3. The mean, median, mode and variance of PLD.

	mean	median	mode	variance
The first data set	4.25595	2.07423	0.0000	139.657
The second data set	1.68412	1.48278	1.16293	1.05093

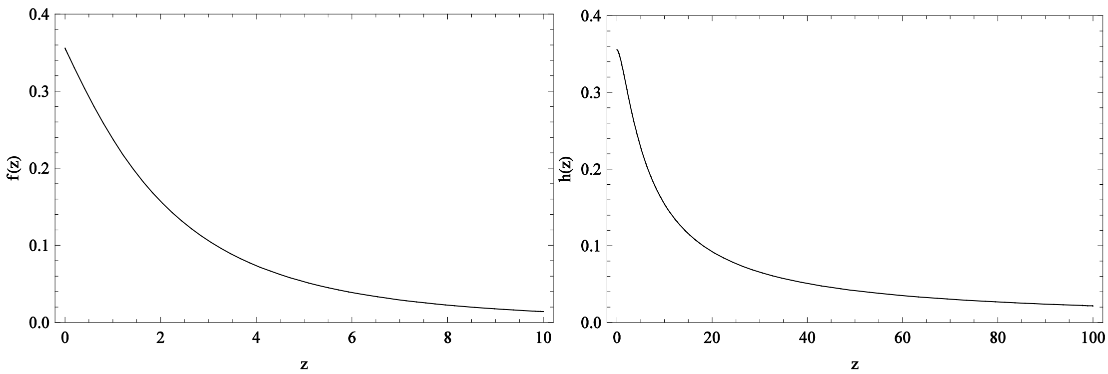

Figure 3. PDF and HRF of PLD of the first data set.

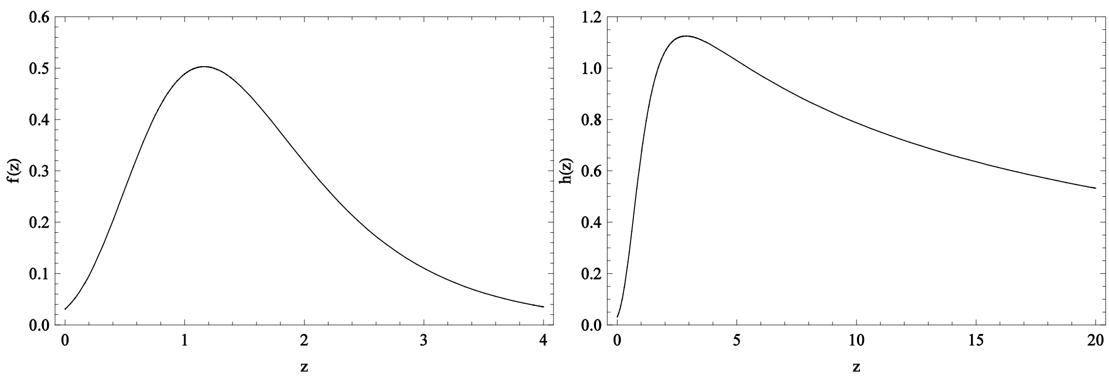

Figure 4. PDF and HRF of PLD of the second data set.

5. Simulation Study

In this section, the ML, UWLS, WLS and Bayes (under LINEX and GE loss functions) estimates of the parameters α, β and γ are computed and compared via a Monte Carlo simulation study as follows:

1 For given values of the prior parameters (μ_1, μ_2, μ_3, σ_1, σ_2), generate values for the parameters (α, β, γ), using Equations (3.6)-(3.8), see AL-Hussaini and Abdel-Hamid (2004) and Abdel-Hamid (2008).

2 Generate a progressively type-II censored sample of size m from PLD (2.5), according to the algorithm given in Balakrishnan and Sandhu (1995).

3 The ML, UWLS, WLS and Bayes (using LINEX and GE loss functions) estimates of the parameters α, β and γ are computed as shown in Section 3.

4 Repeat the above steps $N(=5,000)$ times.

5 If $\hat{\Theta}$ is an estimate of Θ, then the average estimates, mean squared error (MSE) and relative absolute bias (RAB) of $\hat{\Theta}$ over the N samples are given, respectively, by

$$\overline{\hat{\Theta}} = \frac{1}{N}\sum_{i=1}^{N}\hat{\Theta}_i,$$

$$\text{MSE}(\hat{\Theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\Theta}_i - \Theta)^2,$$

$$\text{RAB}(\hat{\Theta}) = \frac{|\overline{\hat{\Theta}} - \Theta|}{\Theta}.$$

6 Calculate the average estimates of the parameters α, β and γ and their MSEs and RABs as shown in Step 5. Calculate also the mean of the MSEs (MMSE) and mean of the RABs (MRAB) according to the following relations:

$$\text{MMSE} = \frac{\text{MSE}(\hat{\alpha}) + \text{MSE}(\hat{\beta}) + \text{MSE}(\hat{\gamma})}{3},$$

$$\text{MRAB} = \frac{\text{RAB}(\hat{\alpha}) + \text{RAB}(\hat{\beta}) + \text{RAB}(\hat{\gamma})}{3}.$$

7 Calculate the CIs of the parameters and then calculate the average interval lengths (AILs) of them. Calculate also the coverage probabilities (COVPs) of the parameters α, β and γ.

The following four CSs are applied in the generation of the samples:

- CS1:

$$R_i = 2, \qquad i = 1, \ldots, \frac{n-m}{2},$$
$$R_i = 0, \qquad \text{otherwise},$$

which means that we remove two units after each observed failure of the first $\frac{n-m}{2}$ failures in the sample.

- CS2:

$$R_i = n-m, \qquad i = 1,$$
$$R_i = 0, \qquad \text{otherwise},$$

which means that we remove $n-m$ units after the first observed failure in the sample.

- CS3:

$$R_i = n-m, \qquad i = \frac{m}{2},$$
$$R_i = 0, \qquad \text{otherwise},$$

which means that we remove $n-m$ units after the middle observed failure in the sample.

- CS4:

$$R_i = n-m, \qquad i = m,$$
$$R_i = 0, \qquad \text{otherwise},$$

which means that we remove $n-m$ units after the last observed failure in the sample.

It may be observed that CS4 is equivalent to traditional type-II censoring.

The values of m have been taken to represent 60%, 80% and 100% of the sample size through the simulation study.

The prior parameters $\mu_1 = 0.851$, $\mu_2 = -3.821$, $\mu_3 = 0.512$, $\sigma_1 = 1.232$ and $\sigma_2 = 0.901$ are considered to generate population parameter values $\alpha = 5.0$, $\beta = 0.5$ and $\gamma = 2.5$ using (3.6)-(3.8).

The computational results are presented in Tables 4 - 6. Table 4 displays the ML, UWLS and WLS estimates of α, β and γ with their MSEs, RABs, MMSE and MRAB, based on $5,000$ simulations, for different values of sample sizes n progressively censored according to four CSs. While Table 5 displays the Bayes estimates of α, β and γ with their MSEs and RABs. The AILs and COVPs of the parameters α, β and γ are displayed in Table 6.

Table 4: ML, UWLS and WLS estimates of α, β and γ with their MSEs, RABs, MMSE and MRAB based on $5,000$ simulations. Population parameter values are $\alpha = 5.0$, $\beta = 0.5$ and $\gamma = 2.5$.

			ML				UWLS				WLS			
n	m	CS	$\bar{\hat{\alpha}}_{ML}$ $\bar{\hat{\beta}}_{ML}$ $\bar{\hat{\gamma}}_{ML}$	MSE($\hat{\alpha}_{ML}$) MSE($\hat{\beta}_{ML}$) MSE($\hat{\gamma}_{ML}$)	RAB($\hat{\alpha}_{ML}$) RAB($\hat{\beta}_{ML}$) RAB($\hat{\gamma}_{ML}$)	MMSE MRAB	$\bar{\hat{\alpha}}_{UW}$ $\bar{\hat{\beta}}_{UW}$ $\bar{\hat{\gamma}}_{UW}$	MSE($\hat{\alpha}_{UW}$) MSE($\hat{\beta}_{UW}$) MSE($\hat{\gamma}_{UW}$)	RAB($\hat{\alpha}_{UW}$) RAB($\hat{\beta}_{UW}$) RAB($\hat{\gamma}_{UW}$)	MMSE MRAB	$\bar{\hat{\alpha}}_{W}$ $\bar{\hat{\beta}}_{W}$ $\bar{\hat{\gamma}}_{W}$	MSE($\hat{\alpha}_{W}$) MSE($\hat{\beta}_{W}$) MSE($\hat{\gamma}_{W}$)	RAB($\hat{\alpha}_{W}$) RAB($\hat{\beta}_{W}$) RAB($\hat{\gamma}_{W}$)	MMSE MRAB
50	30	1	5.8474	5.6472	0.1695	3.1797	5.3726	4.3973	0.0745	2.7724	5.3476	3.9669	0.0695	2.4450
			0.7355	0.5374	0.4710	0.2963	0.7786	0.5739	0.5572	0.2978	0.7364	0.4996	0.4728	0.2572
			3.1207	3.3544	0.2483		3.1542	3.3461	0.2617		3.0732	2.8687	0.2293	
		2	5.6823	5.4889	0.1365	3.2148	5.4006	4.6314	0.0801	2.8323	5.3610	4.3686	0.0722	2.5123
			0.7837	0.6100	0.5674	0.3274	0.7705	0.5515	0.5410	0.2918	0.7229	0.4613	0.4459	0.2442
			3.1961	3.5454	0.2785		3.1359	3.3141	0.2544		3.0362	2.7069	0.2145	
		3	5.7604	5.1329	0.1521	3.1193	5.4249	4.3801	0.0850	2.8795	5.3284	3.8424	0.0657	2.4758
			0.7478	0.5357	0.4956	0.3075	0.7627	0.5627	0.5253	0.2901	0.7279	0.4586	0.4557	0.2538
			3.1875	3.6893	0.2750		3.1502	3.6956	0.2601		3.0997	3.1265	0.2399	
		4	6.2661	6.1872	0.2532	3.7430	6.3536	7.0247	0.2707	3.9032	5.8423	4.2087	0.1685	2.7548
			0.6699	0.5157	0.3398	0.2769	0.6036	0.4430	0.2073	0.2150	0.5914	0.4161	0.1827	0.1598
			3.0943	4.5260	0.2377		2.9176	4.2419	0.1670		2.8205	3.6397	0.1282	
	40	1	5.6588	5.2144	0.1318	2.9255	5.3438	4.2257	0.0688	2.5592	5.2721	3.7253	0.0544	2.2905
			0.7583	0.5410	0.5166	0.3001	0.7679	0.5351	0.5357	0.2825	0.7400	0.4721	0.4800	0.2535
			3.1296	3.0211	0.2518		3.1072	2.9169	0.2429		3.0653	2.6740	0.2261	
		2	5.6488	5.1380	0.1298	2.8815	5.3592	4.3832	0.0718	2.5509	4.8626	4.3674	0.0275	2.3817
			0.7569	0.5284	0.5138	0.2986	0.7510	0.5055	0.5020	0.2632	0.7572	0.4782	0.5144	0.2395
			3.1308	2.9782	0.2523		3.0398	2.7640	0.2159		2.9413	2.2996	0.1765	
		3	5.6966	5.0417	0.1393	2.9240	5.4045	4.6007	0.0809	2.7050	5.2512	3.6173	0.0503	2.3131
			0.7483	0.5306	0.4966	0.2971	0.7594	0.5403	0.5189	0.2784	0.7385	0.4689	0.4778	0.2536
			3.1385	3.1997	0.2554		3.0881	2.9740	0.2353		3.0814	2.8531	0.2326	
		4	5.9026	5.3894	0.1805	3.2313	5.7453	4.9638	0.1491	3.2342	5.5055	3.6728	0.1011	2.4296
			0.7269	0.5450	0.4539	0.2956	0.7177	0.5677	0.4354	0.2758	0.6722	0.4528	0.3444	0.2107
			3.1307	3.7595	0.2523		3.1074	4.1712	0.2430		2.9666	3.1633	0.1866	
	50	—	5.5739	4.8528	0.1148	2.6638	5.3790	4.4680	0.0758	2.4940	5.2463	3.7768	0.0492	2.1816
			0.7494	0.4969	0.4988	0.2838	0.7414	0.4965	0.4828	0.2559	0.7302	0.4464	0.4604	0.2384
			3.0942	2.6418	0.2377		3.0228	2.5175	0.2091		3.0140	2.3216	0.2056	
100	60	1	5.4468	3.3448	0.0894	1.9884	5.2691	3.1444	0.0538	2.0055	5.2439	2.8538	0.0488	1.7693
			0.7020	0.3862	0.4041	0.2315	0.7323	0.4467	0.4645	0.2421	0.6988	0.3689	0.3976	0.2105
			3.0024	2.2343	0.2010		3.0197	2.4255	0.2079		2.9629	2.0852	0.1852	
		2	5.3753	3.5673	0.0751	2.0902	5.3371	3.5240	0.0674	2.0125	5.2689	3.3664	0.0538	1.9439
			0.7314	0.4314	0.4628	0.2515	0.7169	0.4121	0.4338	0.2303	0.7076	0.3940	0.4152	0.2179
			3.0415	2.2721	0.2166		2.9738	2.1013	0.1895		2.9621	2.0714	0.1848	
		3	5.4482	3.2233	0.0896	2.0632	5.3180	3.2131	0.0636	2.1107	5.2041	2.6015	0.0408	1.8446
			0.7089	0.3980	0.4179	0.2432	0.7322	0.4614	0.4643	0.2479	0.7182	0.4139	0.4364	0.2298
			3.0549	2.5684	0.2220		3.0392	2.6574	0.2157		3.0307	2.5184	0.2123	
		4	5.7598	3.6518	0.1520	2.8822	5.8533	4.3748	0.1707	3.1139	5.5300	2.6078	0.1060	2.0379
			0.7159	0.5582	0.4317	0.2808	0.6727	0.4869	0.3454	0.2477	0.6257	0.3865	0.2513	0.1696
			3.1471	4.4366	0.2588		3.0678	4.4798	0.2271		2.8789	3.1194	0.1515	
	80	1	5.3925	3.1577	0.0785	1.6903	5.2706	3.1166	0.0541	1.7721	5.1983	2.6426	0.0397	1.6002
			0.6757	0.3060	0.3514	0.1987	0.7047	0.3743	0.4094	0.2127	0.6887	0.3431	0.3775	0.1951
			2.9158	1.6073	0.1663		2.9369	1.8255	0.1747		2.9201	1.8149	0.1681	
		2	5.3623	3.2655	0.0725	1.7915	5.2659	3.1966	0.0532	1.7749	4.8517	3.3972	0.0297	1.7502
			0.6941	0.3426	0.3882	0.2140	0.6955	0.3652	0.3910	0.2014	0.6993	0.3280	0.3986	0.1856
			2.9534	1.7664	0.1813		2.9001	1.7629	0.1600		2.8212	1.5254	0.1285	
		3	5.3830	3.0951	0.0766	1.8135	5.2527	2.9688	0.0505	1.7716	5.1598	2.4782	0.0320	1.5379
			0.6900	0.3479	0.3800	0.2148	0.7070	0.3859	0.4140	0.2145	0.6877	0.3299	0.3754	0.1926
			2.9698	1.9974	0.1879		2.9478	1.9601	0.1791		2.9258	1.8056	0.1703	
		4	5.5047	3.2627	0.1009	2.2417	5.4747	3.2805	0.0949	2.4388	5.3125	2.4524	0.0625	1.8203
			0.7237	0.4714	0.4473	0.2603	0.7281	0.5160	0.4562	0.2627	0.6829	0.4062	0.3658	0.2037
			3.0816	2.9911	0.2326		3.0928	3.5200	0.2371		2.9570	2.6024	0.1828	
	100	—	5.3414	2.8923	0.0683	1.4784	5.2958	3.1664	0.0592	1.6438	5.2034	2.5576	0.0407	1.3725
			0.6532	0.2580	0.3064	0.1719	0.6737	0.3045	0.3475	0.1840	0.6548	0.2570	0.3096	0.1614
			2.8522	1.2848	0.1409		2.8629	1.4604	0.1452		2.8346	1.3030	0.1338	

Table 4: Continued.

			ML				UWLS				WLS			
n	m	CS	$\bar{\hat{\alpha}}_{ML}$ $\bar{\hat{\beta}}_{ML}$ $\bar{\hat{\gamma}}_{ML}$	MSE($\hat{\alpha}_{ML}$) MSE($\hat{\beta}_{ML}$) MSE($\hat{\gamma}_{ML}$)	RAB($\hat{\alpha}_{ML}$) RAB($\hat{\beta}_{ML}$) RAB($\hat{\gamma}_{ML}$)	MMSE MRAB	$\bar{\hat{\alpha}}_{UW}$ $\bar{\hat{\beta}}_{UW}$ $\bar{\hat{\gamma}}_{UW}$	MSE($\hat{\alpha}_{UW}$) MSE($\hat{\beta}_{UW}$) MSE($\hat{\gamma}_{UW}$)	RAB($\hat{\alpha}_{UW}$) RAB($\hat{\beta}_{UW}$) RAB($\hat{\gamma}_{UW}$)	MMSE MRAB	$\bar{\hat{\alpha}}_{W}$ $\bar{\hat{\beta}}_{W}$ $\bar{\hat{\gamma}}_{W}$	MSE($\hat{\alpha}_{W}$) MSE($\hat{\beta}_{W}$) MSE($\hat{\gamma}_{W}$)	RAB($\hat{\alpha}_{W}$) RAB($\hat{\beta}_{W}$) RAB($\hat{\gamma}_{W}$)	MMSE MRAB
200	120	1	5.1817	1.6384	0.0363	0.9332	5.1741	1.9237	0.0348	1.1522	5.117	1.6636	0.0234	1.0201
			0.6239	0.1764	0.2477	0.1346	0.6538	0.2428	0.3075	0.1588	0.6424	0.2222	0.2848	0.1447
			2.7994	0.9849	0.1197		2.8353	1.2900	0.1341		2.8148	1.1745	0.1259	
		2	5.1903	1.8447	0.0381	0.9906	5.2293	2.3039	0.0459	1.2197	5.1799	2.2401	0.0360	1.2528
			0.6277	0.1872	0.2554	0.1368	0.6442	0.2280	0.2884	0.1522	0.6474	0.2506	0.2949	0.1527
			2.7922	0.9400	0.1169		2.8056	1.1271	0.1222		2.8179	1.2676	0.1272	
		3	5.1593	1.5466	0.0319	1.0679	5.1592	1.9006	0.0318	1.2514	5.0815	1.5259	0.0163	1.0491
			0.6502	0.2363	0.3004	0.1615	0.6700	0.2785	0.3400	0.1753	0.6535	0.2335	0.3070	0.1560
			2.8808	1.4208	0.1523		2.8852	1.5750	0.1541		2.8619	1.3879	0.1448	
		4	5.3648	1.9914	0.0730	2.0595	5.4667	2.4835	0.0933	2.4363	5.2231	1.4754	0.0446	1.6966
			0.7283	0.4816	0.4566	0.2636	0.7101	0.5056	0.4203	0.2543	0.6870	0.4058	0.3740	0.2096
			3.1528	3.7054	0.2611		3.1232	4.3198	0.2493		3.0254	3.2086	0.2102	
	160	1	5.1720	1.4975	0.0344	0.7598	5.1278	1.7743	0.0255	0.9457	5.1070	1.5336	0.0214	0.8518
			0.5971	0.1292	0.1942	0.1062	0.6282	0.1786	0.2563	0.1298	0.6125	0.1660	0.2249	0.1144
			2.7254	0.6527	0.0901		2.7693	0.8842	0.1077		2.7425	0.8557	0.0970	
		2	5.1702	1.5968	0.0340	0.8092	5.2027	2.0023	0.0405	1.0254	5.1244	1.6718	0.0249	0.8908
			0.6041	0.1378	0.2082	0.1126	0.6197	0.1837	0.2394	0.1268	0.6139	0.1664	0.2278	0.1169
			2.7388	0.6930	0.0955		2.7512	0.8903	0.1005		2.7452	0.8343	0.0981	
		3	5.1607	1.4047	0.0321	0.7644	5.1375	1.7416	0.0275	0.9946	5.0820	1.4226	0.0164	0.8525
			0.6016	0.1371	0.2032	0.1115	0.6385	0.2000	0.2770	0.1423	0.6250	0.1742	0.2499	0.1268
			2.7484	0.7515	0.0993		2.8057	1.0422	0.1223		2.7855	0.9608	0.1142	
		4	5.2293	1.7278	0.0459	1.2680	5.2381	1.9922	0.0476	1.7419	5.1259	1.4735	0.0252	1.1992
			0.6679	0.2900	0.3358	0.1844	0.7128	0.4283	0.4256	0.2286	0.6615	0.2971	0.3230	0.1680
			2.9285	1.7861	0.1714		3.0318	2.8052	0.2127		2.8893	1.8271	0.1557	
	200	—	5.1697	1.3638	0.0339	0.6339	5.1947	1.8170	0.0389	0.8907	5.1082	1.4185	0.0217	0.7135
			0.5720	0.0898	0.1440	0.0812	0.5997	0.1458	0.1994	0.1077	0.5932	0.1207	0.1865	0.0965
			2.6642	0.4480	0.0657		2.7121	0.7092	0.0849		2.7036	0.6014	0.0814	
400	240	1	5.0899	0.7361	0.0180	0.3792	5.0984	0.9860	0.0197	0.5322	5.0444	0.8378	0.0089	0.4686
			0.5578	0.0604	0.1156	0.0636	0.5782	0.0974	0.1564	0.0810	0.5771	0.0863	0.1543	0.0771
			2.6433	0.3410	0.0573		2.6674	0.5131	0.0670		2.6703	0.4818	0.0681	
		2	5.1178	0.9342	0.0236	0.4383	5.1289	1.2203	0.0258	0.6075	5.0909	1.1255	0.0182	0.5763
			0.5581	0.0607	0.1161	0.0649	0.5767	0.1000	0.1535	0.0813	0.5739	0.0985	0.1478	0.0766
			2.6379	0.3199	0.0552		2.6617	0.5022	0.0647		2.6594	0.5049	0.0638	
		3	5.0914	0.7270	0.0183	0.3971	5.0701	1.0076	0.0140	0.6072	5.0177	0.8006	0.0035	0.4828
			0.5629	0.0635	0.1257	0.0698	0.5962	0.1211	0.1924	0.0979	0.5860	0.0913	0.1720	0.0857
			2.6634	0.4008	0.0654		2.7182	0.6930	0.0873		2.7038	0.5563	0.0815	
		4	5.1284	1.0296	0.0257	1.3291	5.1881	1.2480	0.0376	1.7502	5.0682	0.7926	0.0136	1.1897
			0.7004	0.3536	0.4007	0.2160	0.7144	0.4401	0.4287	0.2361	0.6690	0.3257	0.3380	0.1785
			3.0539	2.6041	0.2216		3.1047	3.5625	0.2419		2.9593	2.4507	0.1837	
	320	1	5.0868	0.6496	0.0174	0.2958	5.0879	0.9302	0.0175	0.4449	5.0358	0.7419	0.0068	0.3782
			0.5374	0.0367	0.0749	0.0424	0.5595	0.0673	0.1190	0.0617	0.5578	0.0632	0.1157	0.0573
			2.5876	0.2011	0.0350		2.6217	0.3371	0.0487		2.6235	0.3295	0.0494	
		2	5.0806	0.7380	0.0161	0.3289	5.1017	1.0151	0.0203	0.4799	5.0744	0.8669	0.0149	0.4247
			0.5445	0.0399	0.0890	0.0487	0.5629	0.0699	0.1258	0.0664	0.5572	0.0667	0.1144	0.0597
			2.6028	0.2088	0.0411		2.6326	0.3547	0.0530		2.6247	0.3406	0.0499	
		3	5.0839	0.6462	0.0168	0.3073	5.0606	0.9052	0.0121	0.4721	5.0318	0.6707	0.0064	0.3642
			0.5439	0.0396	0.0878	0.0496	0.5764	0.0811	0.1529	0.0771	0.5637	0.0610	0.1273	0.0642
			2.6108	0.2360	0.0443		2.6660	0.4301	0.0664		2.6476	0.3608	0.0590	
		4	5.0763	0.7753	0.0153	0.5286	5.0577	1.0671	0.0196	1.0403	5.0187	0.7535	0.0039	0.5694
			0.5948	0.1126	0.1896	0.1005	0.6638	0.2756	0.3276	0.1696	0.6087	0.1340	0.2175	0.1089
			2.7415	0.6980	0.0966		2.9043	1.7783	0.1617		2.7633	0.8207	0.1053	
	400	—	5.0749	0.5823	0.0150	0.2554	5.0889	0.8375	0.0173	0.3891	5.0472	0.6824	0.0094	0.3231
			0.5330	0.0290	0.0660	0.0372	0.5506	0.0541	0.1012	0.0536	0.5456	0.0449	0.0912	0.0467
			2.5767	0.1548	0.0307		2.6059	0.2756	0.0424		2.5991	0.2420	0.0396	

Table 5: Bayes estimates of α, β and γ with their MSEs and RABs based on 5,000 simulations. Population parameter values are $\alpha = 5.0$, $\beta = 0.5$ and $\gamma = 2.5$. Prior parameter values are $\mu_1 = 0.851$, $\mu_2 = -3.821$, $\mu_3 = 0.512$, $\sigma_1 = 1.232$ and $\sigma_2 = 0.901$.

			LINEX loss function								GE loss function							
			$\nu=-3$				$\nu=-1$				$\nu=-3$				$\nu=-1$			
n	m	CS	$\bar{\hat{\alpha}}_L$ / $\bar{\hat{\beta}}_L$ / $\bar{\hat{\gamma}}_L$	MSE($\hat{\alpha}_L$) / MSE($\hat{\beta}_L$) / MSE($\hat{\gamma}_L$)	RAB($\hat{\alpha}_L$) / RAB($\hat{\beta}_L$) / RAB($\hat{\gamma}_L$)	MMSE / MRAB	$\bar{\hat{\alpha}}_L$ / $\bar{\hat{\beta}}_L$ / $\bar{\hat{\gamma}}_L$	MSE($\hat{\alpha}_L$) / MSE($\hat{\beta}_L$) / MSE($\hat{\gamma}_L$)	RAB($\hat{\alpha}_L$) / RAB($\hat{\beta}_L$) / RAB($\hat{\gamma}_L$)	MMSE / MRAB	$\bar{\hat{\alpha}}_G$ / $\bar{\hat{\beta}}_G$ / $\bar{\hat{\gamma}}_G$	MSE($\hat{\alpha}_G$) / MSE($\hat{\beta}_G$) / MSE($\hat{\gamma}_G$)	RAB($\hat{\alpha}_G$) / RAB($\hat{\beta}_G$) / RAB($\hat{\gamma}_G$)	MMSE / MRAB	$\bar{\hat{\alpha}}_G$ / $\bar{\hat{\beta}}_G$ / $\bar{\hat{\gamma}}_G$	MSE($\hat{\alpha}_G$) / MSE($\hat{\beta}_G$) / MSE($\hat{\gamma}_G$)	RAB($\hat{\alpha}_G$) / RAB($\hat{\beta}_G$) / RAB($\hat{\gamma}_G$)	MMSE / MRAB
50	30	1	4.4227	0.3535	0.1155	0.2248	4.1927	0.7095	0.1615	0.3768	4.1067	0.8681	0.1787	0.4259	3.9901	1.1113	0.2020	0.5266
			0.3749	0.0161	0.2503	0.1929	0.3688	0.0178	0.2624	0.2238	0.3808	0.0145	0.2385	0.2208	0.3662	0.0185	0.2675	0.2442
			1.9680	0.3049	0.2128		1.8808	0.4031	0.2477		1.8869	0.3950	0.2453		1.8423	0.4498	0.2631	
		2	4.4086	0.3690	0.1183	0.2294	4.1678	0.7504	0.1664	0.3907	4.0642	0.9506	0.1872	0.4530	3.9399	1.2277	0.2120	0.5664
			0.3696	0.0176	0.2607	0.1968	0.3630	0.0197	0.2740	0.2290	0.3769	0.0156	0.2462	0.2258	0.3600	0.0206	0.2800	0.2515
			1.9716	0.3017	0.2113		1.8832	0.4020	0.2467		1.8897	0.3928	0.2441		1.8434	0.4509	0.2627	
		3	4.4088	0.3729	0.1182	0.2400	4.1919	0.7142	0.1616	0.3853	4.0858	0.9115	0.1828	0.4506	3.9973	1.0944	0.2005	0.5283
			0.3753	0.0158	0.2493	0.1969	0.3708	0.0171	0.2584	0.2250	0.3808	0.0144	0.2384	0.2257	0.3685	0.0177	0.2631	0.2447
			1.9422	0.3314	0.2231		1.8623	0.4245	0.2551		1.8606	0.4258	0.2557		1.8239	0.4727	0.2704	
		4	4.4008	0.3853	0.1198	0.2570	4.1365	0.8190	0.1727	0.4408	4.0580	0.9648	0.1884	0.4838	3.9265	1.2617	0.2147	0.6076
			0.3694	0.0173	0.2612	0.2056	0.3626	0.0193	0.2748	0.2402	0.3754	0.0157	0.2491	0.2356	0.3600	0.0201	0.2801	0.2615
			1.9103	0.3684	0.2359		1.8171	0.4840	0.2732		1.8268	0.4709	0.2693		1.7761	0.5410	0.2896	
	40	1	4.4275	0.3490	0.1145	0.2117	4.2052	0.6974	0.1590	0.3550	4.1272	0.8371	0.1746	0.4010	4.0120	1.0762	0.1976	0.4958
			0.3796	0.0150	0.2407	0.1844	0.3746	0.0163	0.2508	0.2128	0.3850	0.0136	0.2299	0.2111	0.3722	0.0169	0.2556	0.2324
			2.0048	0.2712	0.1981		1.9285	0.3514	0.2286		1.9278	0.3523	0.2289		1.8902	0.3942	0.2439	
		2	4.4180	0.3600	0.1164	0.2150	4.1927	0.7178	0.1615	0.3633	4.1048	0.8782	0.1790	0.4144	3.9923	1.1184	0.2015	0.5113
			0.3785	0.0153	0.2430	0.1857	0.3736	0.0165	0.2528	0.2148	0.3841	0.0138	0.2318	0.2131	0.3711	0.0172	0.2577	0.2348
			2.0059	0.2698	0.1976		1.9249	0.3556	0.2300		1.9285	0.3512	0.2286		1.8868	0.3984	0.2453	
		3	4.4187	0.3620	0.1163	0.2201	4.2022	0.7032	0.1596	0.3624	4.1162	0.8598	0.1768	0.4135	4.0149	1.0661	0.1970	0.4979
			0.3800	0.0148	0.2399	0.1866	0.3757	0.0159	0.2487	0.2144	0.3850	0.0135	0.2300	0.2138	0.3734	0.0164	0.2531	0.2334
			1.9912	0.2836	0.2035		1.9126	0.3682	0.2350		1.9131	0.3672	0.2347		1.8749	0.4113	0.2500	
		4	4.4028	0.3825	0.1194	0.2347	4.1824	0.7396	0.1635	0.3822	4.0855	0.9189	0.1829	0.4424	3.9900	1.1220	0.2020	0.5248
			0.3760	0.0157	0.2479	0.1933	0.3720	0.0168	0.2560	0.2208	0.3811	0.0144	0.2377	0.2217	0.3697	0.0174	0.2605	0.2402
			1.9681	0.3059	0.2127		1.8927	0.3902	0.2429		1.8890	0.3938	0.2444		1.8548	0.4351	0.2581	
	50	—	4.4318	0.3480	0.1136	0.2002	4.2257	0.6705	0.1549	0.3323	4.1441	0.8150	0.1712	0.3800	4.0469	1.0103	0.1906	0.4590
			0.3861	0.0135	0.2278	0.1750	0.3821	0.0145	0.2359	0.2012	0.3909	0.0124	0.2182	0.2008	0.3799	0.0150	0.2403	0.2196
			2.0409	0.2390	0.1836		1.9683	0.3119	0.2127		1.9674	0.3127	0.2130		1.9305	0.3518	0.2278	
100	60	1	4.5112	0.2639	0.0978	0.1618	4.3770	0.4573	0.1246	0.2497	4.3270	0.5358	0.1346	0.2744	4.2611	0.6456	0.1478	0.3250
			0.3968	0.0114	0.2065	0.1581	0.3930	0.0122	0.2140	0.1792	0.4005	0.0106	0.1989	0.1772	0.3912	0.0126	0.2177	0.1931
			2.0746	0.2100	0.1702		2.0024	0.2795	0.1991		2.0051	0.2768	0.1980		1.9654	0.3169	0.2138	
		2	4.4732	0.3056	0.1054	0.1742	4.3049	0.5625	0.1390	0.2827	4.2439	0.6645	0.1512	0.3153	4.1641	0.8136	0.1672	0.3786
			0.3950	0.0118	0.2100	0.1611	0.3906	0.0127	0.2188	0.1845	0.3991	0.0109	0.2019	0.1827	0.3886	0.0132	0.2228	0.2001
			2.0805	0.2053	0.1678		2.0106	0.2729	0.1957		2.0122	0.2706	0.1951		1.9744	0.3092	0.2102	
		3	4.5064	0.2723	0.0987	0.1726	4.3787	0.4565	0.1243	0.2586	4.3222	0.5464	0.1356	0.2880	4.2650	0.6395	0.1470	0.3328
			0.3947	0.0117	0.2105	0.1636	0.3912	0.0124	0.2176	0.1841	0.3984	0.0109	0.2031	0.1832	0.3894	0.0128	0.2212	0.1979
			2.0464	0.2337	0.1814		1.9735	0.3068	0.2106		1.9731	0.3068	0.2108		1.9359	0.3462	0.2256	
		4	4.4753	0.3148	0.1049	0.2025	4.3159	0.5610	0.1368	0.3138	4.2696	0.6390	0.1461	0.3375	4.1920	0.7744	0.1616	0.3990
			0.3852	0.0137	0.2295	0.1786	0.3809	0.0146	0.2381	0.2029	0.3890	0.0128	0.2220	0.2001	0.3792	0.0151	0.2416	0.2173
			1.9966	0.2789	0.2014		1.9153	0.3657	0.2339		1.9197	0.3607	0.2321		1.8783	0.4076	0.2487	
	80	1	4.5209	0.2576	0.0958	0.1419	4.4017	0.4309	0.1197	0.2178	4.3548	0.5042	0.1290	0.2409	4.2981	0.5976	0.1404	0.2836
			0.4105	0.0090	0.1790	0.1398	0.4069	0.0097	0.1862	0.1581	0.4137	0.0084	0.1727	0.1564	0.4053	0.0100	0.1895	0.1705
			2.1388	0.1590	0.1445		2.0787	0.2128	0.1685		2.0812	0.2101	0.1675		2.0461	0.2431	0.1816	
		2	4.5102	0.2682	0.0980	0.1460	4.3722	0.4739	0.1256	0.2343	4.3299	0.5384	0.1340	0.2532	4.2596	0.6610	0.1481	0.3070
			0.4091	0.0093	0.1817	0.1417	0.4049	0.0101	0.1901	0.1623	0.4124	0.0086	0.1752	0.1593	0.4033	0.0104	0.1935	0.1753
			2.1365	0.1605	0.1454		2.0723	0.2190	0.1711		2.0783	0.2125	0.1687		2.0395	0.2495	0.1842	
		3	4.5193	0.2607	0.0961	0.1491	4.4052	0.4287	0.1190	0.2241	4.3541	0.5070	0.1292	0.2498	4.3049	0.5892	0.1390	0.2886
			0.4075	0.0095	0.1850	0.1449	0.4045	0.0101	0.1910	0.1627	0.4107	0.0089	0.1787	0.1621	0.4029	0.0105	0.1942	0.1750
			2.1160	0.1772	0.1536		2.0548	0.2336	0.1781		2.0543	0.2335	0.1783		2.0208	0.2660	0.1917	
		4	4.4980	0.2888	0.1004	0.1659	4.3752	0.4745	0.1250	0.2472	4.3167	0.5707	0.1367	0.2803	4.2680	0.6512	0.1464	0.3179
			0.4016	0.0107	0.1968	0.1537	0.3989	0.0112	0.2022	0.1719	0.4049	0.0100	0.1903	0.1724	0.3973	0.0115	0.2055	0.1849
			2.0904	0.1981	0.1638		2.0288	0.2560	0.1885		2.0246	0.2602	0.1901		1.9932	0.2910	0.2027	
	100	—	4.5363	0.2428	0.0927	0.1262	4.4386	0.3838	0.1123	0.1863	4.3915	0.4550	0.1217	0.2102	4.3498	0.5227	0.1300	0.2408
			0.4208	0.0074	0.1584	0.1264	0.4189	0.0078	0.1622	0.1403	0.4234	0.0069	0.1532	0.1407	0.4175	0.0080	0.1650	0.1508
			2.1797	0.1283	0.1281		2.1342	0.1674	0.1463		2.1317	0.1688	0.1473		2.1064	0.1917	0.1574	

Table 5: Continued.

			LINEX loss function								GE loss function							
			$\nu = -3$				$\nu = -1$				$\nu = -3$				$\nu = -1$			
			$\hat{\alpha}_L$	MSE($\hat{\alpha}_L$)	RAB($\hat{\alpha}_L$)	MMSE	$\hat{\alpha}_L$	MSE($\hat{\alpha}_L$)	RAB($\hat{\alpha}_L$)	MMSE	$\hat{\alpha}_G$	MSE($\hat{\alpha}_G$)	RAB($\hat{\alpha}_G$)	MMSE	$\hat{\alpha}_G$	MSE($\hat{\alpha}_G$)	RAB($\hat{\alpha}_G$)	MMSE
			$\hat{\beta}_L$	MSE($\hat{\beta}_L$)	RAB($\hat{\beta}_L$)	MRAB	$\hat{\beta}_L$	MSE($\hat{\beta}_L$)	RAB($\hat{\beta}_L$)	MRAB	$\hat{\beta}_G$	MSE($\hat{\beta}_G$)	RAB($\hat{\beta}_G$)	MRAB	$\hat{\beta}_G$	MSE($\hat{\beta}_G$)	RAB($\hat{\beta}_G$)	MRAB
n	m	CS	$\hat{\gamma}_L$	MSE($\hat{\gamma}_L$)	RAB($\hat{\gamma}_L$)		$\hat{\gamma}_L$	MSE($\hat{\gamma}_L$)	RAB($\hat{\gamma}_L$)		$\hat{\gamma}_G$	MSE($\hat{\gamma}_G$)	RAB($\hat{\gamma}_G$)		$\hat{\gamma}_G$	MSE($\hat{\gamma}_G$)	RAB($\hat{\gamma}_G$)	
200	120	1	4.6094	0.1655	0.0781	0.0904	4.5715	0.2166	0.0857	0.1177	4.5540	0.2406	0.0892	0.1253	4.5354	0.2672	0.0929	0.1408
			0.4341	0.0054	0.1317	0.1074	0.4318	0.0058	0.1365	0.1165	0.4360	0.0051	0.1280	0.1148	0.4307	0.0060	0.1386	0.1226
			2.2188	0.1004	0.1125		2.1818	0.1307	0.1273		2.1818	0.1302	0.1273		2.1592	0.1493	0.1363	
		2	4.5753	0.2027	0.0849	0.1028	4.5059	0.3019	0.0988	0.1466	4.4795	0.3400	0.1041	0.1582	4.4455	0.3929	0.1109	0.1833
			0.4330	0.0056	0.1340	0.1103	0.4301	0.0061	0.1399	0.1222	0.4349	0.0052	0.1301	0.1203	0.4290	0.0062	0.1421	0.1299
			2.2198	0.1001	0.1121		2.1805	0.1318	0.1278		2.1834	0.1295	0.1267		2.1579	0.1506	0.1369	
		3	4.6105	0.1644	0.0779	0.0962	4.5775	0.2075	0.0845	0.1237	4.5566	0.2375	0.0887	0.1325	4.5427	0.2549	0.0915	0.1469
			0.4291	0.0062	0.1418	0.1140	0.4265	0.0067	0.1469	0.1239	0.4312	0.0058	0.1376	0.1220	0.4254	0.0069	0.1492	0.1304
			2.1945	0.1180	0.1222		2.1493	0.1571	0.1403		2.1511	0.1542	0.1396		2.1238	0.1789	0.1505	
		4	4.5893	0.1898	0.0821	0.1167	4.5435	0.2553	0.0913	0.1553	4.5143	0.2978	0.0971	0.1683	4.4984	0.3207	0.1003	0.1864
			0.4176	0.0081	0.1648	0.1291	0.4151	0.0086	0.1699	0.1412	0.4201	0.0077	0.1598	0.1396	0.4137	0.0088	0.1725	0.1492
			2.1494	0.1522	0.1403		2.0940	0.2019	0.1624		2.0955	0.1996	0.1618		2.0634	0.2297	0.1746	
	160	1	4.6217	0.1524	0.0757	0.0739	4.5956	0.1878	0.0809	0.0907	4.5811	0.2068	0.0838	0.0971	4.5699	0.2231	0.0860	0.1058
			0.4480	0.0035	0.1039	0.0905	0.4469	0.0036	0.1062	0.0956	0.4491	0.0033	0.1018	0.0953	0.4463	0.0037	0.1074	0.0995
			2.2702	0.0657	0.0919		2.2504	0.0806	0.0998		2.2496	0.0812	0.1002		2.2373	0.0907	0.1051	
		2	4.6125	0.1623	0.0775	0.0771	4.5763	0.2101	0.0847	0.0987	4.5611	0.2327	0.0878	0.1059	4.5419	0.2586	0.0916	0.1183
			0.4474	0.0035	0.1052	0.0916	0.4461	0.0038	0.1078	0.0977	0.4485	0.0034	0.1029	0.0972	0.4454	0.0039	0.1091	0.1023
			2.2696	0.0654	0.0922		2.2481	0.0822	0.1007		2.2479	0.0817	0.1008		2.2346	0.0925	0.1062	
		3	4.6246	0.1494	0.0751	0.0756	4.5980	0.1841	0.0804	0.0935	4.5876	0.1981	0.0825	0.0980	4.5728	0.2184	0.0854	0.1090
			0.4454	0.0038	0.1091	0.0936	0.4441	0.0041	0.1118	0.0995	0.4466	0.0037	0.1067	0.0985	0.4434	0.0042	0.1132	0.1037
			2.2586	0.0735	0.0966		2.2342	0.0922	0.1063		2.2342	0.0923	0.1063		2.2187	0.1044	0.1125	
		4	4.6137	0.1626	0.0773	0.0842	4.5827	0.2042	0.0835	0.1046	4.5662	0.2282	0.0868	0.1137	4.5520	0.2476	0.0896	0.1238
			0.4410	0.0045	0.1181	0.0996	0.4402	0.0047	0.1195	0.1054	0.4423	0.0043	0.1153	0.1057	0.4395	0.0048	0.1211	0.1103
			2.2417	0.0855	0.1033		2.2171	0.1049	0.1132		2.2125	0.1086	0.1150		2.1994	0.1191	0.1202	
	200	—	4.6316	0.1423	0.0737	0.0646	4.6117	0.1688	0.0777	0.0766	4.6041	0.1791	0.0792	0.0798	4.5933	0.1942	0.0813	0.0870
			0.4557	0.0024	0.0886	0.0814	0.4549	0.0026	0.0903	0.0849	0.4563	0.0023	0.0874	0.0844	0.4545	0.0026	0.0910	0.0874
			2.2954	0.0492	0.0818		2.2830	0.0585	0.0868		2.2834	0.0579	0.0867		2.2752	0.0642	0.0899	
400	240	1	4.6501	0.1233	0.0700	0.0554	4.6485	0.1255	0.0703	0.0581	4.6464	0.1280	0.0707	0.0587	4.6465	0.1281	0.0707	0.0602
			0.4597	0.0019	0.0807	0.0758	0.4592	0.0020	0.0816	0.0773	0.4600	0.0019	0.0799	0.0768	0.4590	0.0020	0.0820	0.0782
			2.3082	0.0409	0.0767		2.3003	0.0468	0.0799		2.3008	0.0462	0.0797		2.2954	0.0504	0.0819	
		2	4.6420	0.1327	0.0716	0.0585	4.6373	0.1380	0.0725	0.0610	4.6295	0.1498	0.0741	0.0657	4.6299	0.1477	0.0740	0.0652
			0.4601	0.0019	0.0799	0.0759	0.4602	0.0018	0.0796	0.0767	0.4604	0.0018	0.0792	0.0774	0.4600	0.0019	0.0800	0.0778
			2.3093	0.0409	0.0763		2.3055	0.0432	0.0778		2.3030	0.0455	0.0788		2.3014	0.0460	0.0794	
		3	4.6507	0.1227	0.0699	0.0579	4.6487	0.1254	0.0703	0.0625	4.6480	0.1260	0.0704	0.0615	4.6467	0.1278	0.0707	0.0652
			0.4569	0.0023	0.0862	0.0790	0.4554	0.0025	0.0892	0.0821	0.4574	0.0022	0.0852	0.0803	0.4551	0.0026	0.0898	0.0834
			2.2974	0.0486	0.0810		2.2827	0.0596	0.0869		2.2869	0.0562	0.0853		2.2753	0.0652	0.0899	
		4	4.6481	0.1255	0.0704	0.0645	4.6426	0.1330	0.0715	0.0713	4.6421	0.1330	0.0716	0.0716	4.6383	0.1388	0.0723	0.0763
			0.4504	0.0033	0.0993	0.0867	0.4496	0.0034	0.1008	0.0899	0.4512	0.0031	0.0976	0.0890	0.4491	0.0035	0.1018	0.0920
			2.2737	0.0648	0.0905		2.2567	0.0776	0.0973		2.2557	00787	0.0977		2.2453	0.0866	0.1019	
	320	1	4.6512	0.1223	0.0698	0.0521	4.6499	0.1239	0.0700	0.0532	4.6491	0.1250	0.0702	0.0534	4.6485	0.1257	0.0703	0.0541
			0.4639	0.0014	0.0721	0.0710	0.4637	0.0014	0.0726	0.0716	0.4640	0.0014	0.0719	0.0713	0.4637	0.0014	0.0727	0.0719
			2.3221	0.0326	0.0711		2.3197	0.0344	0.0721		2.3203	0.0338	0.0719		2.3184	0.0353	0.0726	
		2	4.6505	0.1229	0.0699	0.0523	4.6479	0.1263	0.0704	0.0541	4.6476	0.1268	0.0705	0.0541	4.6456	0.1294	0.0709	0.0554
			0.4639	0.0014	0.0723	0.0711	0.4637	0.0014	0.0727	0.0718	0.4640	0.0014	0.0721	0.0715	0.4636	0.0014	0.0728	0.0721
			2.3219	0.0327	0.0712		2.3195	0.0345	0.0722		2.3200	0.0340	0.0720		2.3182	0.0354	0.0727	
		3	4.6516	0.1218	0.0697	0.0527	4.6507	0.1230	0.0699	0.0534	4.6500	0.1239	0.0700	0.0540	4.6496	0.1243	0.0701	0.0543
			0.4631	0.0015	0.0737	0.0719	0.4631	0.0015	0.0737	0.0722	0.4633	0.0015	0.0735	0.0723	0.4630	0.0015	0.0739	0.0726
			2.3191	0.0347	0.0724		2.3172	0.0358	0.0731		2.3163	0.0367	0.0735		2.3153	0.0370	0.0739	
		4	4.6515	0.1217	0.0697	0.0530	4.6491	0.1249	0.0702	0.0551	4.6496	0.1240	0.0701	0.0546	4.6474	0.1271	0.0705	0.0565
			0.4626	0.0015	0.0747	0.0725	0.4621	0.0016	0.0758	0.0736	0.4628	0.0015	0.0744	0.0730	0.4620	0.0016	0.0760	0.0742
			2.3174	0.0358	0.0730		2.3127	0.0389	0.0749		2.3140	0.0382	0.0744		2.3101	0.0408	0.0760	
	400	—	4.6523	0.1211	0.0695	0.0511	4.6519	0.1215	0.0696	0.0513	4.6515	0.1221	0.0697	0.0515	4.6514	0.1221	0.0697	0.0516
			0.4649	0.0013	0.0702	0.0699	0.4649	0.0013	0.0702	0.0700	0.4649	0.0012	0.0702	0.0700	0.4649	0.0013	0.0703	0.0701
			2.3252	0.0308	0.0699		2.3247	0.0312	0.0701		2.3247	0.0311	0.0701		2.3244	0.0314	0.0703	

Table 5: Continued.

			LINEX loss function								GE loss function							
			$\nu=1$				$\nu=3$				$\nu=1$				$\nu=3$			
n	m	CS	$\hat{\alpha}_L$ $\hat{\beta}_L$ $\hat{\gamma}_L$	MSE($\hat{\alpha}_L$) MSE($\hat{\beta}_L$) MSE($\hat{\gamma}_L$)	RAB($\hat{\alpha}_L$) RAB($\hat{\beta}_L$) RAB($\hat{\gamma}_L$)	MMSE MRAB	$\hat{\alpha}_L$ $\hat{\beta}_L$ $\hat{\gamma}_L$	MSE($\hat{\alpha}_L$) MSE($\hat{\beta}_L$) MSE($\hat{\gamma}_L$)	RAB($\hat{\alpha}_L$) RAB($\hat{\beta}_L$) RAB($\hat{\gamma}_L$)	MMSE MRAB	$\hat{\alpha}_G$ $\hat{\beta}_G$ $\hat{\gamma}_G$	MSE($\hat{\alpha}_G$) MSE($\hat{\beta}_G$) MSE($\hat{\gamma}_G$)	RAB($\hat{\alpha}_G$) RAB($\hat{\beta}_G$) RAB($\hat{\gamma}_G$)	MMSE MRAB	$\hat{\alpha}_G$ $\hat{\beta}_G$ $\hat{\gamma}_G$	MSE($\hat{\alpha}_G$) MSE($\hat{\beta}_G$) MSE($\hat{\gamma}_G$)	RAB($\hat{\alpha}_G$) RAB($\hat{\beta}_G$) RAB($\hat{\gamma}_G$)	MMSE MRAB
50	30	1	3.7506	1.6884	0.2499	0.7328	3.0956	3.8247	0.3809	1.4748	3.8626	1.4195	00.2275	0.6469	3.6091	2.1953	0.2782	0.9310
			0.3644	0.0190	0.2711	0.2657	0.3591	0.0206	0.2818	0.3212	0.3484	0.0244	0.3032	0.2694	0.3085	0.0413	0.3831	0.3184
			1.8101	0.4910	0.2760		1.7477	0.5791	0.3009		1.8065	0.4968	0.2774		1.7654	0.5564	0.2938	
		2	3.6268	2.0629	0.2746	0.8590	2.8955	4.6781	0.4209	1.7666	3.7468	1.7504	0.2506	0.7604	3.3790	3.0613	0.3242	1.2326
			0.3568	0.0215	0.2864	0.2789	0.3504	0.0236	0.2991	0.3417	0.3328	0.0306	0.3344	0.2875	0.2707	0.0599	0.4586	0.3604
			1.8108	0.4926	0.2757		1.7372	0.5981	0.3051		1.8062	0.5003	0.2775		1.7544	0.5766	0.2982	
		3	3.7524	1.6761	0.2495	0.7381	3.1125	3.7412	0.3775	1.4593	3.8604	1.4193	0.2279	0.6568	3.6168	2.1423	0.2766	0.9261
			0.3659	0.0184	0.2681	0.2673	0.3612	0.0197	0.2775	0.3219	0.3518	0.0230	0.2965	0.2703	0.3146	0.0381	0.3707	0.3174
			1.7890	0.5198	0.2844		1.7229	0.6169	0.3108		1.7840	0.5282	0.2864		1.7377	0.5978	0.3049	
		4	3.6796	1.8852	0.2641	0.8322	2.9821	4.2820	0.4036	1.6792	3.7865	1.6184	0.2427	0.7516	3.4508	2.7440	0.3098	1.1744
			0.3584	0.0206	0.2832	0.2835	0.3530	0.0222	0.2939	0.3450	0.3389	0.0276	0.3222	0.2907	0.2810	0.0541	0.4380	0.3607
			1.7421	0.5909	0.3032		1.6562	0.7335	0.3375		1.7319	0.6089	0.3072		1.6642	0.7251	0.3343	
	40	1	3.7739	1.6369	0.2452	0.6963	3.1413	3.6504	0.3717	1.3926	3.8833	1.3791	0.2233	0.6125	3.6632	2.0241	0.2674	0.8460
			0.3701	0.0175	0.2598	0.2542	0.3652	0.0188	0.2697	0.3077	0.3567	0.0216	0.2866	0.2560	0.3267	0.0333	0.3467	0.2955
			1.8560	0.4344	0.2576		1.7961	0.5086	0.2816		1.8545	0.4368	0.2582		1.8190	0.4806	0.2724	
		2	3.7374	1.7404	0.2525	0.7313	3.0741	3.9180	0.3852	1.4830	3.8496	1.4679	0.2301	0.6428	3.5974	2.2437	0.2805	0.9215
			0.3690	0.0178	0.2620	0.2575	0.3638	0.0193	0.2725	0.3133	0.3545	0.0224	0.2909	0.2599	0.3190	0.0367	0.3620	0.3052
			1.8549	0.4357	0.2580		1.7944	0.5118	0.2822		1.8533	0.4382	0.2587		1.8172	0.4840	0.2731	
		3	3.7791	1.6192	0.2442	0.6966	3.1681	3.5358	0.3664	1.3602	3.8851	1.3716	0.2230	0.6161	3.6834	1.9411	0.2633	0.8241
			0.3715	0.0170	0.2570	0.2551	0.3669	0.0182	0.2662	0.3065	0.3597	0.0205	0.2805	0.2561	0.3321	0.0310	0.3359	0.2925
			1.8398	0.4535	0.2641		1.7827	0.5267	0.2869		1.8380	0.4562	0.2648		1.8039	0.5002	0.2784	
		4	3.7526	1.6847	0.2495	0.7258	3.1232	3.7100	0.3754	1.4298	3.8586	1.4320	0.2283	0.6443	3.6311	2.1083	0.2738	0.8943
			0.3678	0.0179	0.2643	0.2616	0.3630	0.0193	0.2741	0.3152	0.3546	0.0221	0.2908	0.2636	0.3161	0.0381	0.3678	0.3101
			1.8229	0.4748	0.2709		1.7599	0.5602	0.2960		1.8204	0.4788	0.2718		1.7786	0.5366	0.2886	
	50	—	3.8163	1.5336	0.2367	0.6473	3.2317	3.3070	0.3537	1.2622	3.9203	1.2969	0.2159	0.5698	3.7502	1.7564	0.2500	0.7380
			0.3776	0.0156	0.2449	0.2415	0.3734	0.0167	0.2532	0.2913	0.3666	0.0187	0.2668	0.2419	0.3469	0.0253	0.3061	0.2708
			1.8930	0.3927	0.2428		1.8325	0.4628	0.2670		1.8921	0.3939	0.2431		1.8591	0.4322	0.2564	
100	60	1	4.1101	0.9158	0.1780	0.4283	3.7637	1.6701	0.2473	0.7011	4.1862	0.7799	0.1628	0.3839	4.1111	0.9207	0.1778	0.4417
			0.3898	0.0128	0.2205	0.2090	0.3865	0.0136	0.2270	0.2421	0.3822	0.0146	0.2355	0.2091	0.3730	0.0169	0.2540	0.2240
			1.9283	0.3564	0.2287		1.8698	0.4196	0.2521		1.9278	0.3572	0.2289		1.8992	0.3875	0.2403	
		2	3.9845	1.1742	0.2031	0.5122	3.5512	2.3036	0.2898	0.9105	4.0707	1.0006	0.1859	0.4553	3.9665	1.2412	0.2067	0.5474
			0.3862	0.0137	0.2275	0.2188	0.3831	0.0145	0.2337	0.2577	0.3773	0.0159	0.2453	0.2191	0.3653	0.0194	0.2694	0.2380
			1.9353	0.3487	0.2259		1.8757	0.4136	0.2497		1.9348	0.3495	0.2261		1.9054	0.3816	0.2378	
		3	4.1120	0.9090	0.1776	0.4375	3.7664	1.6526	0.2467	0.7091	4.1876	0.7757	0.1625	0.3939	4.1030	0.9339	0.1794	0.4604
			0.3874	0.0133	0.2253	0.2148	0.3835	0.0142	0.2331	0.2487	0.3802	0.0149	0.2395	0.2146	0.3714	0.0171	0.2572	0.2307
			1.8965	0.3902	0.2414		1.8345	0.4606	0.2662		1.8958	0.3912	0.2417		1.8609	0.4303	0.2556	
		4	4.0509	1.0322	0.1898	0.4969	3.7016	1.8248	0.2597	0.7842	4.1270	0.8921	0.1746	0.4513	4.0391	1.0690	0.1922	0.5242
			0.3781	0.0153	0.2437	0.2314	0.3748	0.0161	0.2503	0.2643	0.3714	0.0170	0.2572	0.2310	0.3633	0.0191	0.2735	0.2465
			1.8484	0.4432	0.2606		1.7931	0.5116	0.2828		1.8473	0.4448	0.2611		1.8151	0.4845	0.2740	
	80	1	4.1648	0.8299	0.1670	0.3712	3.8309	1.5280	0.2338	0.6271	4.2343	0.7092	0.1531	0.3316	4.1602	0.8447	0.1680	0.3899
			0.4036	0.0104	0.1928	0.1847	0.3997	0.0111	0.2005	0.2189	0.3967	0.0118	0.2066	0.1847	0.3871	0.0139	0.2259	0.2011
			2.0143	0.2732	0.1943		1.9443	0.3423	0.2223		2.0140	0.2737	0.1944		1.9762	0.3110	0.2095	
		2	4.1185	0.9111	0.1763	0.3998	3.7543	1.7297	0.2491	0.6961	4.1926	0.7780	0.1615	0.3561	4.1055	0.9524	0.1789	0.4276
			0.4020	0.0107	0.1960	0.1896	0.3985	0.0114	0.2030	0.2254	0.3948	0.0122	0.2104	0.1895	0.3854	0.0143	0.2292	0.2065
			2.0089	0.2777	0.1965		1.9397	0.3472	0.2241		2.0086	0.2781	0.1966		1.9714	0.3160	0.2114	
		3	4.1729	0.8116	0.1654	0.3732	3.8419	1.4973	0.2316	0.6255	4.2416	0.6936	0.1517	0.3344	4.1617	0.8438	0.1677	0.3979
			0.4014	0.0108	0.1972	0.1891	0.3979	0.0115	0.2043	0.2227	0.3947	0.0121	0.2106	0.1890	0.3861	0.0140	0.2278	0.2051
			1.9885	0.2971	0.2046		1.9192	0.3677	0.2323		1.9882	0.2976	0.2047		1.9506	0.3359	0.2198	
		4	4.1283	0.8996	0.1743	0.4126	3.8002	1.5940	0.2400	0.6684	4.1980	0.7764	0.1604	0.3722	4.1223	0.9183	0.1755	0.4334
			0.3959	0.0118	0.2082	0.1997	0.3923	0.0126	0.2155	0.2332	0.3892	0.0133	0.2216	0.1996	0.3806	0.0152	0.2388	0.2155
			1.9584	0.3263	0.2166		1.8895	0.3987	0.2442		1.9580	0.3270	0.2168		1.9198	0.3667	0.2321	
	100	—	4.2303	0.7224	0.1539	0.3173	3.9232	1.3274	0.2154	0.5395	4.2923	0.6195	0.1415	0.2835	4.2233	0.7438	0.1553	0.3359
			0.4157	0.0084	0.1687	0.1644	0.4127	0.0089	0.1746	0.1959	0.4098	0.0095	0.1804	0.1641	0.4017	0.0110	0.1965	0.1789
			2.0738	0.2211	0.1705		2.0061	0.2821	0.1976		2.0738	0.2214	0.1705		2.0380	0.2530	0.1848	

Table 5: Continued.

			LINEX loss function								GE loss function							
			$\nu=1$				$\nu=3$				$\nu=1$				$\nu=3$			
			$\bar{\hat{\alpha}}_L$	MSE($\hat{\alpha}_L$)	RAB($\hat{\alpha}_L$)	MMSE	$\bar{\hat{\alpha}}_L$	MSE($\hat{\alpha}_L$)	RAB($\hat{\alpha}_L$)	MMSE	$\bar{\hat{\alpha}}_G$	MSE($\hat{\alpha}_G$)	RAB($\hat{\alpha}_G$)	MMSE	$\bar{\hat{\alpha}}_G$	MSE($\hat{\alpha}_G$)	RAB($\hat{\alpha}_G$)	MMSE
			$\bar{\hat{\beta}}_L$	MSE($\hat{\beta}_L$)	RAB($\hat{\beta}_L$)	MRAB	$\bar{\hat{\beta}}_L$	MSE($\hat{\beta}_L$)	RAB($\hat{\beta}_L$)	MRAB	$\bar{\hat{\beta}}_G$	MSE($\hat{\beta}_G$)	RAB($\hat{\beta}_G$)	MRAB	$\bar{\hat{\beta}}_G$	MSE($\hat{\beta}_G$)	RAB($\hat{\beta}_G$)	MRAB
n	m	CS	$\bar{\hat{\gamma}}_L$	MSE($\hat{\gamma}_L$)	RAB($\hat{\gamma}_L$)		$\bar{\hat{\gamma}}_L$	MSE($\hat{\gamma}_L$)	RAB($\hat{\gamma}_L$)		$\bar{\hat{\gamma}}_G$	MSE($\hat{\gamma}_G$)	RAB($\hat{\gamma}_G$)		$\bar{\hat{\gamma}}_G$	MSE($\hat{\gamma}_G$)	RAB($\hat{\gamma}_G$)	
200	120	1	4.4917	0.3273	0.1017	0.1678	4.3255	0.5713	0.1349	0.2655	4.5202	0.2879	0.0960	0.1549	4.4888	0.3331	0.1022	0.1785
			0.4300	0.0061	0.1399	0.1293	0.4277	0.0065	0.1446	0.1495	0.4256	0.0069	0.1488	0.1303	0.4194	0.0080	0.1612	0.1403
			2.1343	0.1699	0.1463		2.0776	0.2186	0.1690		2.1344	0.1700	0.1462		2.1064	0.1943	0.1574	
		2	4.3691	0.5107	0.1262	0.2281	4.1388	0.9010	0.1722	0.3732	4.4132	0.4427	0.1174	0.2057	4.3646	0.5197	0.1271	0.2386
			0.4287	0.0063	0.1425	0.1378	0.4269	0.0066	0.1463	0.1615	0.4241	0.0071	0.1518	0.1380	0.4181	0.0081	0.1638	0.1485
			2.1380	0.1672	0.1448		2.0848	0.2120	0.1661		2.1381	0.1674	0.1448		2.1134	0.1881	0.1546	
		3	4.4870	0.3338	0.1026	0.1809	4.3159	0.5808	0.1368	0.2809	4.5162	0.2931	0.0968	0.1677	4.4860	0.3360	0.1028	0.1908
			0.4244	0.0070	0.1512	0.1383	0.4222	0.0074	0.1557	0.1593	0.4196	0.0079	0.1608	0.1395	0.4132	0.0090	0.1736	0.1498
			2.0973	0.2018	0.1611		2.0368	0.2545	0.1853		2.0973	0.2020	0.1611		2.0676	0.2275	0.1730	
		4	4.4207	0.4382	0.1159	0.2365	4.2407	0.7057	0.1519	0.3454	4.4567	0.3850	0.1087	0.2193	4.4342	0.4136	0.1132	0.2386
			0.4114	0.0092	0.1772	0.1607	0.4099	0.0096	0.1803	0.1818	0.4058	0.0103	0.1883	0.1620	0.4001	0.0114	0.1997	0.1711
			2.0273	0.2621	0.1891		1.9669	0.3209	0.2132		2.0271	0.2626	0.1891		1.9986	0.2908	0.2005	
	160	1	4.5307	0.2775	0.0939	0.1282	4.3964	0.4709	0.1207	0.2017	4.5522	0.2478	0.0896	0.1185	4.5256	0.2854	0.0949	0.1352
			0.4456	0.0039	0.1088	0.1047	0.4444	0.0040	0.1112	0.1193	0.4429	0.0043	0.1142	0.1050	0.4391	0.0049	0.1219	0.1115
			2.2216	0.1032	0.1114		2.1854	0.1303	0.1259		2.2217	0.1033	0.1113		2.2054	0.1152	0.1178	
		2	4.4924	0.3308	0.1015	0.1466	4.3456	0.5462	0.1309	0.2276	4.5197	0.2918	0.0961	0.1338	4.4979	0.3223	0.1004	0.1480
			0.4447	0.0040	0.1107	0.1084	0.4439	0.0041	0.1122	0.1234	0.4418	0.0044	0.1165	0.1085	0.4384	0.0049	0.1232	0.1142
			2.2178	0.1051	0.1129		2.1820	0.1324	0.1272		2.2179	0.1051	0.1128		2.2027	0.1167	0.1189	
		3	4.5345	0.2733	0.0931	0.1324	4.4116	0.4447	0.1177	0.1991	4.5552	0.2447	0.0890	0.1230	4.5362	0.2711	0.0928	0.1358
			0.4425	0.0043	0.1149	0.1093	0.4418	0.0044	0.1164	0.1229	0.4396	0.0048	0.1209	0.1099	0.4361	0.0053	0.1277	0.1155
			2.2003	0.1196	0.1199		2.1633	0.1481	0.1347		2.2004	0.1196	0.1198		2.1854	0.1311	0.1259	
		4	4.5063	0.3144	0.0987	0.1520	4.3697	0.5141	0.1261	0.2304	4.5307	0.2797	0.0939	0.1407	4.5070	0.3156	0.0986	0.1581
			0.4383	0.0049	0.1233	0.1171	0.4374	0.0051	0.1252	0.1327	0.4349	0.0055	0.1302	0.1178	0.4310	0.0062	0.1380	0.1245
			2.1770	0.1368	0.1292		2.1333	0.1721	0.1467		2.1771	0.1369	0.1292		2.1581	0.1524	0.1368	
	200	—	4.5704	0.2254	0.0859	0.0988	4.4737	0.3575	0.1053	0.1498	4.5852	0.2055	0.0830	0.0923	4.5682	0.2288	0.0864	0.1039
			0.4547	0.0026	0.0907	0.0897	0.4531	0.0028	0.0938	0.1010	0.4531	0.0028	0.0938	0.0897	0.4498	0.0033	0.1004	0.0951
			2.2689	0.0685	0.0924		2.2404	0.0892	0.1038		2.2690	0.0685	0.0924		2.2538	0.0797	0.0985	
400	240	1	4.6440	0.1304	0.0712	0.0616	4.6278	0.1499	0.0744	0.0719	4.6459	0.1282	0.0708	0.0609	4.6431	0.1319	0.0714	0.0641
			0.4593	0.0020	0.0814	0.0785	0.4586	0.0021	0.0827	0.0822	0.4584	0.0021	0.0832	0.0790	0.4568	0.0023	0.0864	0.0813
			2.2924	0.0525	0.0831		2.2760	0.0638	0.0896		2.2925	0.0524	0.0830		2.2844	0.0581	0.0862	
		2	4.6116	0.1740	0.0777	0.0756	4.5712	0.2234	0.0858	0.0950	4.6188	0.1646	0.0762	0.0725	4.6172	0.1647	0.0766	0.0738
			0.4595	0.0019	0.0809	0.0803	0.4591	0.0020	0.0817	0.0849	0.4586	0.0021	0.0827	0.0804	0.4574	0.0022	0.0853	0.0820
			2.2946	0.0508	0.0822		2.2818	0.0597	0.0873		2.2946	0.0508	0.0821		2.2894	0.0546	0.0842	
		3	4.6444	0.1300	0.0711	0.0676	4.6289	0.1484	0.0742	0.0788	4.6462	0.1279	0.0708	0.0670	4.6435	0.1323	0.0713	0.0709
			0.4551	0.0026	0.0898	0.0845	0.4546	0.0027	0.0909	0.0886	0.4537	0.0028	0.0926	0.0853	0.4520	0.0031	0.0961	0.0878
			2.2684	0.0702	0.0926		2.2483	0.0853	0.1007		2.2685	0.0702	0.0926		2.2597	0.0772	0.0961	
		4	4.6330	0.1446	0.0734	0.0813	4.6064	0.1766	0.0787	0.1017	4.6368	0.1397	0.0726	0.0798	4.6339	0.1433	0.0732	0.0865
			0.4488	0.0035	0.1025	0.0943	0.4469	0.0038	0.1061	0.1021	0.4466	0.0038	0.1068	0.0954	0.4428	0.0045	0.1145	0.1007
			2.2328	0.0959	0.1069		2.1963	0.1248	0.1215		2.2329	0.0959	0.1068		2.2137	0.1117	0.1145	
	320	1	4.6468	0.1279	0.0706	0.0551	4.6371	0.1398	0.0726	0.0601	4.6479	0.1265	0.0704	0.0546	4.6462	0.1288	0.0708	0.0560
			0.4637	0.0014	0.0727	0.0721	0.4635	0.0014	0.0731	0.0736	0.4634	0.0014	0.0732	0.0722	0.4629	0.0015	0.0742	0.0730
			2.3172	0.0359	0.0731		2.3124	0.0392	0.0750		2.3172	0.0359	0.0731		2.3149	0.0376	0.0740	
		2	4.6423	0.1334	0.0715	0.0572	4.6289	0.1495	0.0742	0.0634	4.6443	0.1309	0.0711	0.0564	4.6429	0.1328	0.0714	0.0573
			0.4634	0.0014	0.0732	0.0728	0.4634	0.0014	0.0731	0.0741	0.4631	0.0015	0.0738	0.0728	0.4629	0.0015	0.0742	0.0732
			2.3159	0.0369	0.0736		2.3122	0.0393	0.0751		2.3160	0.0369	0.0736		2.3148	0.0377	0.0741	
		3	4.6486	0.1254	0.0703	0.0558	4.6418	0.1336	0.0716	0.0596	4.6495	0.1244	0.0701	0.0554	4.6485	0.1256	0.0703	0.0562
			0.4626	0.0016	0.0748	0.0735	0.4626	0.0016	0.0748	0.0747	0.4622	0.0016	0.0755	0.0737	0.4618	0.0017	0.0763	0.0743
			2.3112	0.0403	0.0755		2.3057	0.0437	0.0777		2.3112	0.0403	0.0755		2.3093	0.0414	0.0763	
		4	4.6453	0.1298	0.0709	0.0580	4.6340	0.1428	0.0732	0.0640	4.6467	0.1280	0.0707	0.0574	4.6454	0.1292	0.0709	0.0585
			0.4620	0.0016	0.0761	0.0747	0.4619	0.0016	0.0763	0.0766	0.4615	0.0017	0.0771	0.0749	0.4608	0.0018	0.0783	0.0758
			2.3074	0.0425	0.0770		2.2995	0.0476	0.0802		2.3075	0.0425	0.0770		2.3043	0.0445	0.0783	
	400	—	4.6507	0.1230	0.0699	0.0521	4.6469	0.1275	0.0706	0.0538	4.6511	0.1225	0.0698	0.0519	4.6505	0.1234	0.0699	0.0523
			0.4648	0.0013	0.0704	0.0703	0.4648	0.0013	0.0704	0.0707	0.4647	0.0013	0.0706	0.0703	0.4646	0.0013	0.0707	0.0704
			2.3237	0.0319	0.0705		2.3224	0.0325	0.0710		2.3237	0.0319	0.0705		2.3232	0.0321	0.0707	

Table 6. The AILs and COVPs (in %) of 95% CIs based on 5,000 simulations. Population parameter values are $\alpha = 5.0$, $\beta = 0.5$ and $\gamma = 2.5$.

n	m	CS	AIL(α) AIL(β) AIL(γ)	COVP(α) COVP(β) COVP(γ)	n	m	CS	AIL(α) AIL(β) AIL(γ)	COVP(α) COVP(β) COVP(γ)
50	30	1	21.7507	99.82	200	120	1	5.0189	96.74
			10.4198	92.04				1.3816	92.48
			19.4330	95.18				3.4864	94.74
		2	14.2456	99.76			2	6.2181	97.00
			4.3977	93.20				1.5873	93.22
			10.4918	96.02				3.8213	95.56
		3	14.588	99.88			3	6.4869	96.74
			5.0039	93.36				1.9936	92.94
			12.5146	95.98				5.1787	95.28
		4	19.1301	99.98			4	10.0402	99.80
			8.5262	93.00				5.2211	91.36
			25.129	94.76				15.2101	93.12
	40	1	13.9177	99.36		160	1	4.7277	96.44
			3.6797	92.82				1.1857	93.08
			9.3560	95.91				2.8138	95.17
		2	14.1188	99.48			2	4.9623	96.5
			3.9328	93.47				1.2204	92.91
			9.8096	96.05				2.8655	95.27
		3	13.1804	99.57			3	5.0704	96.63
			4.7863	93.42				1.4078	93.58
			11.1822	95.77				3.4432	95.51
		4	16.1841	99.97			4	6.1900	98.52
			6.1030	93.51				2.3255	91.82
			16.1849	95.12				0.2276	94.18
	50	—	10.9213	99.03		200	—	4.4847	96.50
			2.7108	93.79				1.0214	93.48
			6.9397	96.37				2.3667	95.25
100	60	1	10.3797	98.62	400	240	1	3.4978	95.92
			3.0817	92.02				0.8975	93.60
			7.9613	95.12				2.1757	95.60
		2	11.2446	98.68			2	4.3833	96.18
			3.2731	92.30				1.0435	93.32
			8.0826	95.62				2.4188	95.34
		3	8.8418	98.82			3	3.2085	96.08
			2.8254	92.40				0.8604	93.64
			7.7228	95.48				2.1987	95.62
		4	14.7265	99.96			4	5.7666	98.50
			6.9757	91.48				2.8757	92.24
			20.3918	93.34				8.3788	94.02
	80	1	8.4715	98.01		320	1	3.2479	95.66
			2.2096	91.81				0.7365	93.74
			5.5625	95.17				1.7363	95.02
		2	7.8955	97.80			2	3.2287	95.56
			2.0419	92.18				0.7094	94.38
			5.1599	95.20				1.6546	95.94
		3	8.3141	98.12			3	3.2707	95.68
			2.3040	92.19				0.8114	94.30
			5.9926	95.16				1.9703	95.38
		4	9.4746	99.74			4	3.4465	96.06
			3.5346	92.15				1.1192	93.30
			9.6164	94.51				2.9045	94.80
	100	—	8.2206	97.32		400	—	3.1886	95.90
			2.1143	92.00				0.6814	94.62
			5.0834	94.82				1.5879	95.62

1. *Numerical Results*

From the numerical results carried out via simulation studies, it can be observed that the Bayes estimates (using LINEX and GE loss functions) are better than the ML, UWLS and WLS estimates via the MMSEs and MRABs. For all $\nu < 0$, it can be also observed that the Bayes estimates using LINEX loss function are better than those using GE loss function via the MMSEs and MRABs. It can be also observed that the WLS estimates are better than the UWLS estimates via the MMSEs and MRABs. By increasing the sample size n, we have observed that the MLEs are better than the UWLS and WLS estimates via the MMSEs and MRABs.

- From Tables 4 - 5 the following points can be observed:

1. For fixed values of n, by increasing m, the MSEs, RABs, MMSEs and MRABs decrease.
2. For fixed values of m, by increasing n, the MSEs, RABs, MMSEs and MRABs decrease.

- From Table 5 it can be observed that, for fixed values of n and m, by decreasing υ the MSEs, RABs, MMSEs and MRABs decrease.
- From Table 6 the following points can be observed:

1. The COVPs are closer to the nominal value (95%) by increasing n except in some rare cases, this may be due to fluctuation in the data.
2. For fixed values of n, by increasing m, the AILs decrease.
3. For fixed values of m, by increasing n, the AILs decrease.

Furthermore, it should be pointed out that if the hyper-parameters are unknown, the empirical Bayes method to estimate them using past samples may be used, see Maritz and Lwin (1989). Alternatively, the hierarchical Bayes method could be used in which a suitable prior for the hyper-parameters is used, see Bernardo and Smith (1994).

Remark 5.1.

1. Another simulation study based on other population parameter values has been performed from which results near to the results given in this simulation study have been obtained.

2. The integrals presented in Subsection 3.3 may be obtained using Legendre-Gauss quadrature formula, Canuto et al. (2006), or subroutine "qand " in IMSL subroutines.

3. The mathematica software and IMSL subroutines have been used in the computations.

6. Concluding Remarks

In this paper, we have considered the PLD distribution, with decreasing and upside down shapes of failure rate, as a lifetime distribution under CR model. Two real data sets have been used to compare among PLD, ELD, EEPD, EEGD, EWPD and LD which have showed that the PLD is better to fit the data than the other five distributions.

Among the motivations of the PLD are:

1. It could be applied in the CR model and parallel systems.
2. The PLD has decreasing and upside down shapes of failure rate which make it suitable to fit several real data.
3. Better to fit the data than some other distributions such as ELD, EEPD, EEGD, EWPD and LD.

Based on progressive type-II censoring, we have discussed five estimation methods to estimate the parameters α, β and γ. The methods that have been discussed are ML, UWLS, WLS and Bayes (using LINEX and GE loss functions) estimations. The performance of these methods has been investigated through a simulation study, based on four different progressive CSs.

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They also dedicate the paper to the spirit of late Professor Essam K. AL-Hussaini who passed away on Aug., 2015.

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