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Groups with Fuzzy Operations (F-groups)

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Abstract

In this paper, we are going to develop the notion of groups with fuzzy operations (**F**-groups). An **F**-group is a set equipped with a fuzzy equality and a binary fuzzy operation. In the rest, we investigate the notions of fuzzy subgroup, fuzzy normal subgroup and isomorphism theorems in **F**-groups setting.

Keywords: **F**-algebra, **F**-group, Fuzzy subgroup, Fuzzy normal subgroup, Isomorphism theorems.

Introduction

The problem of development of algebras with fuzzy operations is formulated in ([1], P: 136). Fuzzy approaches to various universal algebraic concepts started with Rosenfeld's fuzzy groups [10]. Since then, many fuzzy algebraic structures have been studied (vector spaces, rings, etc.). Also, some authors proposed a general approach to the theory of fuzzy algebras. Another fuzzy approach to universal algebras was initiated by Belohlvek and Vychodil [1,2], who studied the so-called algebras with fuzzy equalities and developed fuzzy equational logic. These structures have two parts: the functional part, which is an ordinary algebra and the relational part, which is the carrier set of the algebra, equipped with a fuzzy equality which is compatible with all of the fundamental operations of the ordering algebra. In the fuzzy set theory there were many different approaches to the concept of a fuzzy function. In a number of papers various kinds of fuzzy functions based on fuzzy equivalence relations were studied. In particular, such approach was used in definitions of partial fuzzy functions and fuzzy functions, given by Klawonn [9], strong fuzzy functions and perfect fuzzy functions, given by Demirci [5,6]. Fuzzy functions based on fuzzy equivalence relations have shown oneself to be very useful in many applications in approximate

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reasoning, fuzzy control, vague algebra and other fields. The content of this paper can be briefly stated as follows: In the present section, we introduce the necessary information concerned with the fuzzy operation and algebras with fuzzy operations, and in the next section, we define **F**- groups by fuzzy operations. In the rest of this paper, we define **F**- congruences on **F**-groups, **F**- quotient groups by **F**- congruences and isomorphism theorems with their properties.

In this paper we will use complete residuated lattices $\boldsymbol{L} =< L,\wedge,\vee,\otimes,\rightarrow,0,1 >$ as the structures of truth values. Residuated lattices were introduced by Ward and Dilworth in ring theory. Complete residuated lattices as a structures of truth values were introduced into the context of fuzzy sets and fuzzy logic by Goguen [3].

A complete residuated lattice is an algebra $\boldsymbol{L} =< L,\wedge,\vee,\otimes,\rightarrow,0,1 >$ where

(i) $< L,\wedge,\vee,0,1 >$ is a complete lattice with the least element 0 and the greatest element 1,

(ii) $< L,\otimes,1 >$ is a commutative monoid, i.e. $\otimes$ is associative, commutative, and $a\otimes 1 = a$ for each $a \in L$,

(iii) $\otimes$, and $\rightarrow$ satisfy adjointness, i.e. $a\otimes b \le c$ iff $a \le b \rightarrow c$ for each $a,b,c \in L$ ($\le$ denotes the lattice ordering).

The operations $\otimes$ (called multiplication) and $\rightarrow$ (residuum) are intended for modeling the conjunction and implication of the corresponding logical calculus, while supremum ($\vee$) and infimum ($\wedge$) are intended for modeling of the existential ($\exists$) and universal ($\forall$) quantifer.

An **L**-set of X (or fuzzy set with truth degrees in L) is a mapping $\mu: X \rightarrow L$. An n-ary **L**-relation of X is a mapping $\mu: X^n \rightarrow L$.

Definition 1.1. [1] A fuzzy equivalence relation E on a set X is a mapping $E: X\times X \rightarrow L$ satisfying

(i) $E(x,y) = 1$ (Reflexivity);

(ii) $E(x,y) = E(y,x)$ (Symmetry);

(iii) $E(x,y) \otimes \mathrm{E(y,z)} \le E(x,z)$ (Transitivity),

for every $x,y,z \in X$. A fuzzy equivalence E on X where E(x, y)=1 implies $x = y$ will be called a fuzzy equality. Fuzzy equalities will usually be denoted by $\approx$.

Theorem 1.2. [1] Each complete residuated lattice satisfies

$$a \otimes \textstyle\bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \otimes b_i), \tag{1}$$

$$a_1 \le a_2 \ \text{ and } \ b_1 \le b_2 \ \text{ implies } \ a_1 \otimes a_2 \le b_1 \otimes b_2. \tag{2}$$

Theorem 1.3. Let $\approx^{\boldsymbol{M}}$ be a fuzzy equality on *M*. Let $(a_1,\dots,a_n)\approx^{M^n}(b_1,\dots,b_n)$=$\otimes_{i=1}^{n}(a_i\approx^{M}b_i)$. Then $\approx^{M^n}$ is a fuzzy equality on M^n.

Proof. Reflexivity: $(a_1,\dots,a_n)\approx^{M^n}(a_1,\dots,a_n)$=$\otimes_{i=1}^{n}(a_i\approx^{M}a_i)=1$.

Symmetry:

$(a_1,\dots,a_n)\approx^{M^n}(b_1,\dots,b_n)=\otimes_{i=1}^{n}(a_i\approx^{M}b_i)=\otimes_{i=1}^{n}(b_i\approx^{M}a_i)=(b_1,\dots,b_n)\approx^{M^n}(a_1,\dots,a_n)$.

Transitivity: Using (2) we have

$$\begin{aligned}(a_1,\dots,a_n)\approx^{M^n}(b_1,\dots,b_n)\otimes(b_1,\dots,b_n)\approx^{M^n}(c_1,\dots,c_n)=\\ =[\otimes_{i=1}^{n}(a_i\approx^{M}b_i)]\otimes[\otimes_{i=1}^{n}(b_i\approx^{M}c_i)]=\\ =\otimes_{i=1}^{n}\left((a_i\approx^{M}b_i)\otimes(b_i\approx^{M}c_i)\right)\le\\ \le\otimes_{i=1}^{n}(a_i\approx^{M}c_i)=(a_1,\dots,a_n)\approx^{M^n}(c_1,\dots,c_n).\end{aligned}$$

If $(a_1,\dots,a_n)\approx^{M^n}(b_1,\dots,b_n)=1$ then $\otimes_{i=1}^{n}(a_i\approx^{M}b_i)=1$, hence $a_i=b_i$ for all $i\in\{1,2,\dots,n\}$

Definition 1.4. Let $\approx^{\boldsymbol{M}}$ be a fuzzy equality on *M*. A fuzzy relation $r\colon M^n\times M\to L$ is called an n-ary fuzzy operation w.r.t. $\approx^{\boldsymbol{M}}$ if we have the following conditions

Extensionality:

$$\otimes_{i=1}^{n+1}(a_i\approx^{M}b_i)\otimes r(a_1,\dots,a_{n+1})\le r(b_1,\dots,b_{n+1}),\ \forall a_i,\ \forall b_i\in M, i\in\{1,..,n+1\},$$

Functionality:

$$r(a_1,\dots,a_n,y)\otimes r(a_1,\dots,a_n,y')\le(y\approx^{M}y'),\qquad \forall a_i,\forall y,\ \forall y'\in M, i\in\{1,\dots,n\},$$

Fully defined: $\bigvee_{y\in M}r(a_1,\dots,a_n,y)=1,\forall a_1,\dots,\forall a_n\in M$.

From Extensionality, it is simply proved that

$$(x_i\approx^{M}y_i)\otimes r(x_1,\dots,x_i,..,x_{n+1})\le r(x_1,\dots,y_i,..,x_{n+1}). \tag{3}$$

Throughout this paper, we will use the above inequality.

Definition 1.5. [4] Let $\tau=<\approx,R>$ be a type where $\approx\notin R$ and each $r\in R$ is called a relation symbol, $\approx$ is a binary relation symbol called a symbol for fuzzy equality. Then an algebra with fuzzy operations is a triplet $\boldsymbol{M}=<M,\approx^{\boldsymbol{M}},R^{\boldsymbol{M}}>$ such that

(i) $\approx^{\mathbf{M}}$ is a fuzzy equality on the set M,

(ii) $R^{\mathbf{M}}$ is a set of fuzzy operations on the set M.

To simply, we call **F**-algebra instead of an algebra with fuzzy operations.

In [6] a strong fuzzy function is introduced by Demirci, but we develop this notion to algebras. An **F**-algebra $\boldsymbol{M}=<M,\approx^{\boldsymbol{M}},R^{\boldsymbol{M}}>$ is called a strong **F**-algebra if for every $x_1,\dots,x_n\in M,\exists\, y\in M$ such that $r^M(x_1, \dots, x_n, y) = 1$, where $r^M \in R^M$ is an arbitrary n-ary fuzzy operation on M.

Results and Discusion

Definition 2.1. Let $\boldsymbol{G} =< G, \approx^{\boldsymbol{G}}, *^{\boldsymbol{G}} >$ be an **F**-algebra of type $\tau =<\approx, *>$, where $*^{\mathbf{G}}$ is a binary fuzzy operation on G. Then

(i) $*^{\boldsymbol{G}}$ is a fuzzy abelian iff $*^{\boldsymbol{G}}$ satisfies

$$(\forall a, \forall b, \forall c, \forall c' \in G)\ [(*^{\boldsymbol{G}}\,(a, b, c) \otimes *^{\boldsymbol{G}}\,(b, a, c')) \leq (c \approx^{\boldsymbol{G}} c')].$$

(ii) $*^{\boldsymbol{G}}$ is a fuzzy associative iff $*^{\boldsymbol{G}}$ satisfies

$$\left[(*^{\boldsymbol{G}}\,(b, c, d) \otimes *^{\boldsymbol{G}}\,(a, d, m) \otimes *^{\boldsymbol{G}}\,(a, b, q) \otimes *^{\boldsymbol{G}}\,(q, c, w)) \leq (m \approx^{\boldsymbol{G}} w)\right]$$

for all $a, b, c, d, m, q, w \in G$.

Definition 2.2. Let $\boldsymbol{G} =< G, \approx^{\boldsymbol{G}}, *^{\boldsymbol{G}} >$ be a strong F-algebra of type $\tau =<\approx, *>$. Then $\boldsymbol{G} =< G, \approx^{\boldsymbol{G}}, *^{\boldsymbol{G}} >$ is an **F**-group if $*^{\boldsymbol{G}}$ is fuzzy associative and we have the following conditions

(i) There exists an (two sided) identity element $e \in G$ such that

$$*^{\boldsymbol{G}}\,(a, e, a) = *^{\boldsymbol{G}}\,(e, a, a) = 1 \text{ for each } a \in G.$$

(ii) For a given identity element $e \in G$, and for a given $a \in G$, there exists an element

$$a^{-1} \in G \text{ such that } *^{G}\,(a^{-1}, a, e) = *^{G}\,(a, a^{-1}, e) = 1.$$

To simply, we denote the F-group $\boldsymbol{G} =< G, \approx^{\boldsymbol{G}}, *^{\boldsymbol{G}} >$ by $\boldsymbol{G} =< G, \approx, * >$.

Example 2.3. Let $\mathbb{Z}_3 =< \{[0], [1], [2]\}, + >$ be the group with the integers (mod 3) with addition.

Then $\mathbb{Z}'_3 =< \{[0], [1], [2]\}, \approx', +' >$ with the following fuzzy equality $\approx'$ on $\mathbb{Z}_3$

$$\mathrm{x} \approx' \mathrm{y} = \begin{cases} 1, & \text{if } \mathrm{x} = \mathrm{y}, \\ 0, & \text{if } \mathrm{x} \neq \mathrm{y}, \end{cases} \quad \forall x, y \in \mathbb{Z}_3,$$

and with the fuzzy operation $+'(x, y, z) = (x + y) \approx' z$ on $\mathbb{Z}_3$ is an **F**-group.

Theorem 2.4. Let $\boldsymbol{G} =< G, \approx, * >$ be an **F**-group. Then

(i) The identity element e is an unique element.

(ii) For every $a \in G$, $a^{-1} \in G$ is an unique inverse element.

Proof. (i): Let e and e' be two identity elements of G. Using the condition (ii) of Definition 2.2 and the functionality of *, we have

$1 = *\,(e, e', e) \otimes *\,(e, e', e') \leq (e \approx e')$, i.e. $\mathrm{e} \approx \mathrm{e}' = 1$. Sine $\approx$ is a fuzzy equality thus, $e = e'$.

(ii): Let $\mathrm{a} \in \mathrm{G}$ has two inverse elements b and b'. By the associativity of * and the conditions (i), (ii) for **F**-groups, we have

$1 = *\,(a, b'e) \otimes *\,(b, e, b) \otimes *\,(b, a, e) \otimes *\,(e, b', b') \leq (b \approx b')$, *i.e.* $b \approx b' = 1$, hence $b = b'$.

In [8], vague groups and the cancellation law for vague groups are introduced by Demrci, Since **F**-groups are similar to the vague groups, thus we introduce the cancellation law for **F**-groups.

Theorem 2.5. [8]. (Cancellation Law) Let $\boldsymbol{G} =< G, \approx, * >$ be an **F**-group. Then

(i) $*(a,b,c) \otimes *(a,b^{'},c) \leq (b \approx b^{'})$,

(ii) $*(a,b,c) \otimes *(a^{\prime},b,c) \leq (a \approx a^{'})$,

for every $a,b,b',c \in G$.

Theorem 2.6. Let $\boldsymbol{G} =< G, \approx, * >$ be an **F**-group and e the identity element of G.

Then

(i) $*(a,b,e) \leq (b \approx a^{-1})$,

(ii) $*(a,b,e) \leq (a \approx b^{-1})$,

(iii) $*(c,c,c) \leq (c \approx e)$,

where $a,b,c \in G$, and a^{-1}, b^{-1} are the inverses of a, b, respectively.

Proof. (i): By the cancellation law we have

$$*(a,b,e) =*(a,b,e) \otimes 1 =*(a,b,e) \otimes *(a,a^{-1},e) \leq (b \approx a^{-1}).$$

(ii) is similarly proved.

(iii) By the cancellation law we may write

$*(c,c,c) = 1 \otimes *(c,c,c) =*(c,e,c) \otimes *(c,c,c) \leq (c \approx e)$, for every $c \in G$.

In the rest, we define and investigate the notions of subgroup, congruence, and morphism on **F**-groups. In [3], a fuzzy subalgebra in an ordinary algebra $\boldsymbol{M} =< M, F^{\boldsymbol{M}} >$ is introduced by an **L**-set A in M such that for each n-ary operation $f^{\boldsymbol{M}} \in F^{\boldsymbol{M}}$ we have $A(a_1), \dots, A(a_n) \leq A(f^{\boldsymbol{M}}(a_1, \dots, a_n))$ for every $a_1, \dots, a_n \in M$.

Definition 2.7. Let $\boldsymbol{G}$ be any F-group. By a fuzzy subgroup μ of $\boldsymbol{G}$ is defined as a function $\mu: G \to L$ such that

$$\mu(a) \otimes \mu(b) \leq \vee_{c \in G}(\mu(c) \otimes *(a, b^{-1}, c))$$

for all $a,b \in G$. Moreover a fuzzy normal subgroup μ of $\boldsymbol{G}$ is defined as a fuzzy subgroup satisfying the condition

$$\vee_{c \in G}\big(\mu(c) \otimes *(a,b,c)\big) \leq \vee_{c^{'} \in G}\big(\mu(c^{'}) \otimes *(b,a,c^{'})\big)$$

for every $a,b \in G$.

In the following, we get into the concept of morphisms for **F**-groups.

Definition 2.8. Let $\boldsymbol{G} =< G, \approx, * >$ and $\boldsymbol{G}^{'} =< G^{'}, \approx^{'}, *^{'} >$ be two F-groups. A mapping $f: G \to G'$ is called a morphism of G to G' if

(i) $a \approx b \le f(a) \approx' f(b)\ \forall a, \forall b \in G,$

(ii) $*\,(a_1, a_2, a_3) \le *'\,(f(a_1), f(a_2), f(a_3)), \forall a_1,\ \forall a_2,\ \forall a_3 \in G.$

The fact that $f: G \to G'$ is a morphism is denoted by $f: \boldsymbol{G} \to \boldsymbol{G}'$. Furthermore,

(a) a morphism such that

$$(a \approx b) = \big(f(a) \approx' f(b)\big)\ \forall a, \forall b \in G$$

and

$$*\,(\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3) = *'\big(f(\mathrm{a}_1), \mathrm{f}(\mathrm{a}_2), \mathrm{f}(\mathrm{a}_3)\big),$$

is called an embedding, for every $a_1, a_2, a_3 \in G$.

(b) a surjective morphism is called an epimorphism,

(c) an injective morphism is called a monomorphism,

(d) an epimorphism which is an embedding is called an isomorphism.

(e) an epimorphism with the condition $(a \approx b) = \big(f(a) \approx' f(b)\big)$ is called a weak isomorphism, where $a, b \in G$.

Theorem 2.9. Let $\boldsymbol{G} =< G, \approx, * >$ and $\boldsymbol{G}' =< G', \approx', *' >$ be two **F**-groups and $f: \boldsymbol{G} \to \boldsymbol{G}'$ be a morphism. Then

(i) $f(e) = e'$ where e and e' are the identity elements of G and G' respectively.

(ii) $f(b^{-1}) = \big(f(b)\big)^{-1}$, where b^{-1} is the inverse element of any $b \in G$.

(iii) $f(b) \approx' e' \le \big(f(b)\big)^{-1} \approx' e',\ \forall b \in G.$

Proof. To prove (i), from the cancellation law for **F-**groups and $1 = (e, e, e) \le *'\,\big(f(e), f(e), f(e)\big)$, thus $1 = *'\,\big(\mathrm{f(e), f(e), f(e)}\big)$, we have

$1 = 1 \otimes 1 = *'\,\big(f(e), e', f(e)\big) \otimes *'\,\big(f(e), f(e), f(e)\big) \le \big(e' \approx' f(e)\big)$. This implies that $\mathrm{e}' \approx' \mathrm{f(e)} = 1$, i.e. $\mathrm{e}' = \mathrm{f(e)}$.

(ii): By the condition (ii) of **F**-groups we have $1 = *\,(b, b^{-1}, e) \le *'\big(f(b), f(b^{-1}), f(e)\big)$, thus $*'\big(f(b), f(b^{-1}), f(e)\big) = 1$. So, using the cancellation law it is proved that

$$1 = 1 \otimes 1 = *'\,\big(f(b), f(b^{-1}), f(e)\big) \otimes *'\,\Big(f(b), \big(f(b)\big)^{-1}, e'\Big) =$$

$$= *'\,(\mathrm{f(b), f(b^{-1}), e'}) \otimes *'\,\Big(\mathrm{f(b)}, \big(\mathrm{f(b)}\big)^{-1}, \mathrm{e}'\Big) \le \mathrm{f(b^{-1})} \approx' \big(\mathrm{f(b)}\big)^{-1},$$

therefore $f(b^{-1}) \approx' \big(f(b)\big)^{-1} = 1$. This means that $f(b^{-1}) = \big(f(b)\big)^{-1}$.

(iii): Using the cancellation law, the condition (ii) of **F**-groups, and (3), we have

$$f(b) \approx' e' = (f(b) \approx' e') \otimes 1 = (f(b) \approx' e') \otimes *' \left(f(b), \left(f(b)\right)^{-1}, e'\right) \leq *' \left(e', \left(f(b)\right)^{-1}, e'\right)$$

$$= *' \left(e', \left(f(b)\right)^{-1}, e'\right) \otimes 1 = *' \left(e', \left(f(b)\right)^{-1}, e'\right) \otimes *' (e', e', e') \leq (\left(f(b)\right)^{-1} \approx' e').$$

Definition 2.10. Let $\boldsymbol{G} =< G, \approx, * >$ and $\boldsymbol{G'} =< G', \approx ', * ' >$ be two **F**-groups and $f: \boldsymbol{G} \to \boldsymbol{G'}$ be a morphism. For any fuzzy subgroup μ of $\boldsymbol{G'}$ we define a map $f^{-1}(\mu)$ from G to L by

$$f^{-1}(\mu)(x) = \mu(f(x))$$

for all $x \in G$, we call a preimage of fuzzy subgroup μ under f. For any subgroup μ of $\boldsymbol{G}$ we define an image $f[\mu]$ of μ under f by from G' to L by $f[\mu](y) = \bigvee_{u \in f^{-1}(y)} \mu(u)$ for all $y \in G'$.

Theorem 2.11. Let $\boldsymbol{G} =< G, \approx, * >$ and $\boldsymbol{G'} =< G', \approx ', * ' >$ be two **F**-groups, $f: \boldsymbol{G} \to \boldsymbol{G}'$ be an embedding. Let μ' be a fuzzy (normal) subgroup of $\boldsymbol{G'}$, then the preimage $f^{-1}(\mu')$ is a fuzzy (normal) subgroup of $\boldsymbol{G}$.

Proof. First, we show that $f^{-1}(\mu')$ is a fuzzy subgroup of $\boldsymbol{G}$. Due to the definition of subgroups for **F**-groups and since f is an embedding, then we have

$f^{-1}(\mu')(a) \otimes f^{-1}(\mu')(b) = \mu'(f(a)) \otimes \mu'(f(b)) \leq$

$\bigvee_{c' \in G'}(\mu'(c') \otimes *' (f(a), \left(f(b)\right)^{-1}, c')) \leq \bigvee_{c \in G}(\mu'(f(c)) \otimes *' (f(a), f(b^{-1}), f(c)))$

(since $c' \in G', \exists c \in G$ such that $f(c) = c'$)

$= \bigvee_{c \in G}(f^{-1}(\mu')(c) \otimes * (a, b^{-1}, c))$.

Thus, $f^{-1}(\mu')$ is a fuzzy subgroup of $\boldsymbol{G}$.

For being fuzzy normal subgroup of $f^{-1}(\mu')$ on $\boldsymbol{G}$, by the definition of fuzzy normal subgroups, we have to prove $\bigvee_{c \in G}(f^{-1}(\mu')(c) \otimes * (a, b, c)) \leq \bigvee_{t \in G}(f^{-1}(\mu')(t) \otimes * (b, a, t))$.

So, we have

$\bigvee_{c \in G}(f^{-1}(\mu')(c) \otimes * (a, b, c)) = \bigvee_{c \in G}(\mu'(f(c)) \otimes *'(f(a), f(b), f(c)))$

$= \bigvee_{c' \in G'}(\mu'(c') \otimes *'(f(a), f(b), c'))$ (where $f(c) = c'$)

$\leq \bigvee_{c'' \in G'}(\mu'(c'') \otimes *'(f(b), f(a), c''))$ (since μ' is a fuzzy normal subgroup of $\boldsymbol{G'}$)

$\leq \bigvee_{t \in G}(\mu'(f(t)) \otimes *'(f(b), f(a), f(t)))$ (since $c'' \in G'$ implies $\exists\, t \in G$ such that $f(t) = c''$)

$= \bigvee_{t \in G}(f^{-1}(\mu')(t) \otimes * (b, a, t))$.

Theorem 2.12. Let $\boldsymbol{G} =< G, \approx, * >$ and $\boldsymbol{G'} =< G', \approx ', * ' >$ be two **F**-groups, $f: \boldsymbol{G} \to \boldsymbol{G}'$ be an embedding and monomorphism. Let μ be a fuzzy (normal) subgroup of $\boldsymbol{G}$, then the image $f[\mu]$ is a fuzzy (normal) subgroup of $\boldsymbol{G}'$.

Proof. Since f is monomorphism, then for every $c,d \in G'$ there exist unique elements $a,b \in G$ such that $f(a)=c, f(b)=d$ thus $f[\mu](c)=\mu(a)$, $f[\mu](d)=\mu(b)$.

$$f[\mu](c) \otimes f[\mu](d) = \mu(a) \otimes \mu(b)$$

$$\leq \vee_{m\in G} \left(\mu(m) \otimes * (a, b^{-1}, m)\right) \text{ (since } \mu \text{ is a fuzzy subgroup of } \mathbf{G})$$

$$= \vee_{m\in G} \left(\mu(m) \otimes *' \left(f(a), f(b^{-1}), f(m)\right)\right)$$

$$= \vee_{m'\in G'} \left(f[\mu](m') \otimes *' (c, d^{-1}, m')\right).$$

where $f(m)=m'$. Thus, $f[\mu]$ is a fuzzy subgroup of $\boldsymbol{G}'$.

By the previous reason, for every $a',b',c' \in G'$ there exist unique elements $a,b,c \in G$ such that $f(a)=a', f(b)=b', f(c)=c'$.

$$\vee_{c'\in G'} f[\mu](c') \otimes *' (a',b',c') = \vee_{c\in G} \mu(c) \otimes *' (f(a), f(b), f(c)$$

$$= \vee_{c\in G} \mu(c) \otimes * (a,b,c)$$

$$\leq \vee_{c''\in G} \mu(c'') \otimes * (b,a,c'') \text{ (since } \mu \text{ is a fuzzy normal subgroup of } \mathbf{G})$$

$$= \vee_{c''\in G} \mu(c'') \otimes *' (f(b), f(a), f(c''))$$

$$= \vee_{t\in G'} f[\mu](t) \otimes *' (b', a', t)$$

where $f(c'')=t$. Hence, $f[\mu]$ is a fuzzy normal subgroup of $\boldsymbol{G}'$.

Defenition 2.13. Let $\boldsymbol{G} = < G, \approx, * >$ be an **F**-group. A binary **L**-relation (binary fuzzy relation) θ on G is called an **F**-congruence on G if

(i) θ is a fuzzy equivalence on G,

(ii) θ is compatible with $\approx$, i.e. $(a \approx b) \otimes (a' \approx b') \otimes \theta(a,a') \leq \theta(b,b')$ for every $a,a',b,b' \in G$,

(iii) $\left(\otimes_{i=1}^{3} \theta(a_i,b_i)\right) \otimes * (a_1,a_2,a_3) \leq * (b_1,b_2,b_3)$ for every $a_1,a_2,a_3,b_1,b_2,b_3 \in G$.

Remark 2.14. The condition (ii) in Definition 2.13 holds iff $\approx \subseteq \theta$. [3, Lemma 1.82].

Definition 2.15. Let θ be an **F**-congruence on an **F**-group $\boldsymbol{G} = < G, \approx, * >$. An **F**-quotient group G by θ is an **F**-group $\boldsymbol{G}/\theta = < G/\theta, \approx^{\boldsymbol{G}/\theta}, *^{\boldsymbol{G}/\theta} >$ such that

(i) $[a]_\theta \approx^{\boldsymbol{G}/\theta} [b]_\theta = \theta(a,b) \ \forall \, [a]_\theta, [b]_\theta \in G/\theta$;

(ii) $*^{\boldsymbol{G}/\theta}([a_1]_\theta, [a_2]_\theta, [a_3]_\theta) = \vee_{c\in G} \left(* (a_1,a_2,c) \otimes \theta(c,a_3)\right)$, where $G/\theta = \{[a]_\theta \mid a \in G\}$ and $[a]_\theta = \{a' \in G \mid \theta(a,a') = 1\}$.

Remark 2.16. An **F**-quotient group is well-defined. First, it is clear $\approx^{\boldsymbol{G}/\theta}$ is a well-defined. Second, $*^{\boldsymbol{G}/\theta}$ is a well-defined fuzzy operation on G/θ which is extensional w.r.t. $\approx^{\boldsymbol{G}/\theta}$.

$$\left[\otimes_{i=1}^{3} \left([a_i]_\theta \approx^{\boldsymbol{G}/\theta} [b_i]_\theta\right)\right] \otimes *^{\boldsymbol{G}/\theta} ([a_1]_\theta, [a_2]_\theta, [a_3]_\theta) =$$

$$= \left(\otimes_{i=1}^{3} \theta(a_i,b_i)\right) \otimes \vee_{c\in G} \left(* (a_1,a_2,c) \otimes \theta(c,a_3)\right)$$

$$= \theta(a_1, b_1) \otimes \theta(a_2, b_2) \otimes \theta(a_3, b_3) \otimes \vee_{c\in G} \left(* (a_1, a_2, c) \otimes \theta(c, a_3)\right)$$

$$= \vee_{c\in G} \Big(\theta(a_1, b_1) \otimes \theta(a_2, b_2) \otimes \theta(a_3, b_3) \otimes \left(* (a_1, a_2, c) \otimes \theta(c, a_3)\right)\Big)$$

(by the inequality (1))

$$\leq \vee_{c\in G} (* (b_1, b_2, c) \otimes \theta(c, b_3)) = *^{\boldsymbol{G}/\theta} ([b_1]_\theta, [b_2]_\theta, [b_3]_\theta).$$

Now, we prove that $\approx^{\mathbf{G}/\theta}$ satisfies in the functionality condition.

$$*^{\boldsymbol{G}/\theta} ([a_1]_\theta, [a_2]_\theta, [a_3]_\theta) \otimes *^{\boldsymbol{G}/\theta} ([a_1]_\theta, [a_2]_\theta, [b_3]_\theta) =$$

$$= \vee_{c\in G} \left(* (a_1, a_2, c) \otimes \theta(c, a_3)\right) \otimes \vee_{c'\in G} (* (a_1, a_2, c') \otimes \theta(c', b_3))$$

$$= \vee_{c,c'\in G} [(* (a_1, a_2, c)) \otimes \theta(c, a_3) \otimes (* (a_1, a_2, c')) \otimes \theta(c', b_3)]$$

$$\leq * (a_1, a_2, a_3) \otimes * (a_1, a_2, b_3) \leq (a_3 \approx b_3) \leq \theta(a_3, b_3) = \left([a_3]_\theta \approx^{\boldsymbol{G}/\theta} [b_3]_\theta\right).$$

Finally, it is simply proved that $\boldsymbol{G}/\theta$ is an **F**-group.

Definition 2.17. Let $\boldsymbol{G}$ and $\boldsymbol{G}'$ be two **F**-groups. let $h: \boldsymbol{G} \to \boldsymbol{G}'$ be a morphism. Then the kernel of $h,$ the binary **L**-set (fuzzy set) $\theta_h : G \times G \to L,$ is defined by

$$\theta_h(a, b) = h(a) \approx' h(b).$$

Theorem 2.18. Let $\boldsymbol{G}$ and $\boldsymbol{G}'$ be two **F**-groups. let $h: \boldsymbol{G} \to \boldsymbol{G}'$ be an embedding. Then θ_h is an **F**-congruence on $\boldsymbol{G}$.

Proof. It is clear that θ_h is a fuzzy equivalence on G. Now, for every $a, b, c, d \in G$ we have

$$(a \approx b) \otimes (c \approx d) \otimes \theta_h(a, c) = \left(h(a) \approx' h(b)\right) \otimes \Big(\left(h(c) \approx' h(d)\right) \otimes (h(a) \approx' h(c)\Big) \leq$$

$h(b) \approx' h(d) = \theta_h(b, d)$. *i.e.* θ_h is compatible w.r.t. $\approx$.

$$\left[\otimes_{i=1}^{3} \theta_h(a_i, b_i)\right] \otimes * (a_1, a_2, a_3) = \left[\otimes_{i=1}^{3} (h(a_i) \approx' h(b_i))\right] \otimes * (a_1, a_2, a_3)$$

$$= \left[\otimes_{i=1}^{3} (a_i \approx b_i)\right] \otimes * (a_1, a_2, a_3) \leq * (b_1, b_2, b_3),$$

for every $a_1, a_2, a_3 \in G$. Altogether, θ_h is an **F**-congruence on $\boldsymbol{G}$.

Definition 2.19. For every **F**-group $\boldsymbol{G}$ and θ an **F**-congruence on $\boldsymbol{G}$, a mapping $h_\theta: \boldsymbol{G} \to \boldsymbol{G}/\theta$ where $h_\theta(a) = [a]_\theta$ for all $a \in G$ is called a natural mapping.

Theorem 2.20. A natural mapping $h_\theta: \boldsymbol{G} \to \boldsymbol{G}/\theta$ is an epimorphism.

Proof. For any $a, b \in G$ we have

$$a \approx^{\boldsymbol{G}} b \leq \theta(a, b) = [a]_\theta \approx^{\boldsymbol{G}/\theta} [b]_\theta = h_\theta(a) \approx^{\boldsymbol{G}/\theta} h_\theta(b).$$

Furthermore, for every $a_1, a_2, a_3 \in G$ we have

$$*^{\boldsymbol{G}} (a_1, a_2, a_3) = *^{\boldsymbol{G}} (a_1, a_2, a_3) \otimes \theta(a_3, a_3) \leq \vee_{c\in G} \left(*^{\boldsymbol{G}} (a_1, a_2, c) \otimes \theta(c, a_3)\right)$$

$$= *^{\boldsymbol{G}/\theta} ([a_1]_\theta, [a_2]_\theta, [a_3]_\theta)$$

$$=*^{\boldsymbol{G}/\theta}\ \big(h_\theta(a_1), h_\theta(a_2), h_\theta(a_3)\big).$$

Surjectivity of h_θ is evident.

Theorem 2.21. (**first isomorphism theorem**). Let $h\colon \boldsymbol{G} \to \boldsymbol{G}'$ be an embedding of **F**-groups. Then there is an isomorphism $g\colon \boldsymbol{G} / \theta_h \to \boldsymbol{G}'$ such that $h_{\theta_h} \circ g = h$.

Proof. Let $g\colon G/\theta_h \to G'$ be a mapping with $g([a]_{\theta_h}) = h(a)$ for all $a \in G$.

Evidently, $h_{\theta_h} \circ g = h$. Furthermore, we have

$$[a]_{\theta_h} \approx^{\boldsymbol{G}/\theta_h} [b]_{\theta_h} = \theta_h(a,b) = h(a) \approx' h(b) = g([a]_{\theta_h}) \approx' g([b]_{\theta_h}).$$

From the surjectivity of h, it follws that g is surjective.

To check the condition (ii) for morphisms of **F**-groups, we have

$$*^{\boldsymbol{G}/\theta_h}\ \big([a_1]_{\theta_h}, [a_2]_{\theta_h}, [a_3]_{\theta_h}\big) =$$
$$=\vee_{c\in G}\ \big(*\ (a_1, a_2, c) \otimes \theta_h(c, a_3)\big)$$
$$=\vee_{c\in G}\ \big(*\ (a_1, a_2, c) \otimes (h(c) \approx' h(a_3))\big)$$
$$=\vee_{c\in G}\ (*\ (a_1, a_2, c) \otimes (c \approx a_3)\ \text{ (because } h \text{ is an embedding)}$$
$$\leq *\ (\mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3)\ \text{ (by the inequality (3))}$$
$$=*'\ (h(a_1), h(a_2), h(a_3))$$
$$=*'\big(g([a_1]_{\theta_h}), g([a_2]_{\theta_h}), g([a_3]_{\theta_h})\big),$$

where $[a_1]_{\theta_h}, [a_2]_{\theta_h}, [a_3]_{\theta_h} \in (G/\theta_h)$. This implies that

$$*^{G/\theta_h}\ \big([a_1]_{\theta_h}, [a_2]_{\theta_h}, [a_3]_{\theta_h}\big) \leq *'\big(g([a_1]_{\theta_h}), g([a_2]_{\theta_h}), g([a_3]_{\theta_h})\big). \tag{4}$$

On the other hand, since θ_h is an **F**-congruence on $\boldsymbol{G}$, then by Theorem 2.20, $h_{\theta_h}\colon \boldsymbol{G} \to \boldsymbol{G}/\theta_h$ is an epimorphism on $\boldsymbol{G}/\theta_h$. Thus, $*\ (a_1, a_2, a_3) \leq *^{\boldsymbol{G}/\theta_h}\ (h_{\theta_h}(a_1), h_{\theta_h}(a_2), h_{\theta_h}(a_3))$. Hence, we have

$$*'\big(g([a_1]_{\theta_h}), g([a_2]_{\theta_h}), g([a_3]_{\theta_h})\big) = *'\ (h(a_1), h(a_2), h(a_3))$$
$$=*\ (a_1, a_2, a_3)$$
$$\leq *^{\boldsymbol{G}/\theta_h}\ (h_{\theta_h}(a_1), h_{\theta_h}(a_2), h_{\theta_h}(a_3))$$
$$=*^{\boldsymbol{G}/\theta_h}\ \big([a_1]_{\theta_h}, [a_2]_{\theta_h}, [a_3]_{\theta_h}\big).$$

Therefore,

$$*'\big(g([a_1]_{\theta_h}), g([a_2]_{\theta_h}), g([a_3]_{\theta_h})\big) \leq *^{\boldsymbol{G}/\theta_h}\ \big([a_1]_{\theta_h}, [a_2]_{\theta_h}, [a_3]_{\theta_h}\big). \tag{5}$$

By (4) and (5), $*^{\boldsymbol{G}/\theta_h}\ \big([a_1]_{\theta_h}, [a_2]_{\theta_h}, [a_3]_{\theta_h}\big) = *'\big(g([a_1]_{\theta_h}), g([a_2]_{\theta_h}), g([a_3]_{\theta_h})\big)$.

This means that g is isomorphism.

Definition 2.22. Let φ, θ be two **F**-congruences of an **F**-group $\boldsymbol{G}$ and $\theta \subseteq \varphi$ Then we let φ/θ denote a binary **L**-relation (binary fuzzy relation) on $(\boldsymbol{G} / \theta)$ defined by $(\varphi/\theta)([a]_\theta, [b]_\theta) = \varphi(a,b)$

for all $a, b \in G$.

Theorem 2.23. Let φ, θ be two **F**-congruences of an **F**-group $\boldsymbol{G}$ and $\theta \subseteq \varphi$ Then φ/θ is an **F**-congruence of $(\boldsymbol{G} / \theta)$.

Proof. Clearly, φ/θ is a fuzzy equivalence on $(\boldsymbol{G}/\theta)$. For all $a, b \in G$ we have

$[a]_\theta \approx^{\boldsymbol{G}/\theta} [a]_\theta = \theta(a,b) \leq \varphi(a,b) = (\varphi/\theta)([a]_\theta, [b]_\theta)$,

Therefore $\approx^{\mathbf{G}/\theta} \subseteq \varphi/\theta$. Due to Remark 2.14, φ/θ is compatible w.r.t. $\approx^{\boldsymbol{G}/\theta}$.

To check condition (iii) in the definition of **F**-congruences, for arbitrary elements $[a_1]_\theta, [b_1]_\theta, [a_2]_\theta, [b_2]_\theta, [a_3]_\theta, [b_3]_\theta \in \mathrm{G}/\theta$, we have

$$\left[\otimes_{i=1}^3 (\varphi/\theta)([a_i]_\theta, [b_i]_\theta)\right] \otimes *^{\mathbf{G}/\theta} ([a_1]_\theta, [a_2]_\theta, [a_3]_\theta) =$$
$$= \left[\otimes_{i=1}^3 \varphi(a_i, b_i)\right] \otimes \bigvee_{c\in \mathrm{G}} (*\,(a_1, a_2, c) \otimes \theta(c, a_3))$$
$$\leq \left[\otimes_{i=1}^3 \varphi(a_i, b_i)\right] \otimes \bigvee_{c\in \mathrm{G}} (*\,(a_1, a_2, c) \otimes \varphi(c, a_3))$$
$$= \bigvee_{c\in \mathrm{G}} \left[\otimes_{i=1}^3 \varphi(a_i, b_i) \otimes *\,(a_1, a_2, c) \otimes \varphi(c, a_3)\right] \text{ (by the inequality (1))}$$
$$= \bigvee_{c\in \mathrm{G}} [\varphi(a_1, b_1) \otimes \varphi(a_2, b_2) \otimes \varphi(a_3, b_3) \otimes *\,(a_1, a_2, c) \otimes \varphi(c, a_3)]$$
$$\leq \bigvee_{c\in \mathrm{G}} [\varphi(a_1, b_1) \otimes \varphi(a_2, b_2) \otimes \varphi(c, b_3) \otimes *\,(a_1, a_2, c)]$$
$$\leq \bigvee_{c\in \mathrm{G}} (*\,(b_1, b_2, c) \otimes \varphi(c, b_3)) = *^{\mathbf{G}/\theta} ([b_1]_\theta, [b_2]_\theta, [b_3]_\theta).$$

Therefore, φ/θ is an **F**-congruence on $(\boldsymbol{G} / \theta)$.

Theorem 2.24. (**second isomorphism theorem**). Suppose $\boldsymbol{G}$ is an **F**-group and φ, θ are two **F**-congruences on $\boldsymbol{G}$ and $\theta \subseteq \varphi$. Then the mapping

$$h: (\mathbf{G} / \theta) / (\varphi/\theta) \longrightarrow \mathbf{G} / \varphi$$

defined by $h([[a]_\theta]_{\varphi/\theta}) = [a]_\varphi$ is a weak isomorphism.

Proof. For every $[[a]_\theta]_{\varphi/\theta}, [[b]_\theta]_{\varphi/\theta} \in (G / \theta) / (\varphi/\theta)$ we have

$[[a]_\theta]_{\varphi/\theta} \approx^{(\boldsymbol{G}/\theta)/(\varphi/\theta)} [[b]_\theta]_{\varphi/\theta} = (\varphi/\theta)([a]_\theta, [b]_\theta) = \varphi(a,b) = [a]_\varphi \approx^{\boldsymbol{G}/\varphi} [b]_\varphi$.

Since for all $a \in \mathrm{G}$ we have $h([[a]_\theta]_{\varphi/\theta}) = [a]_\varphi$, h is surjective.

Also, we suppose $[[a]_\theta]_{\varphi/\theta}, [[b]_\theta]_{\varphi/\theta}, [[c]_\theta]_{\varphi/\theta} \in (\mathrm{G}/\theta)/(\varphi/\theta)$, then

$*^{(\mathbf{G}/\theta)/(\varphi/\theta)} \left([[a]_\theta]_{\varphi/\theta}, [[b]_\theta]_{\varphi/\theta}, [[c]_\theta]_{\varphi/\theta}\right) =$

$$= \bigvee_{[e]_\theta \in \mathrm{G}/\theta} \left[*^{\mathbf{G}/\theta} ([a]_\theta, [b]_\theta, [e]_\theta) \otimes (\varphi/\theta)([e]_\theta, [c]_\theta)\right]$$
$$= \bigvee_{[e]_\theta \in \mathrm{G}/\theta} [\bigvee_{e' \in \mathrm{G}} (*\,(a, b, e') \otimes \theta(e', e)) \otimes \varphi(e, c)]$$
$$\leq \bigvee_{[e]_\theta \in \mathrm{G}/\theta} [\bigvee_{e' \in \mathrm{G}} (*\,(a, b, e') \otimes \varphi(e', e)) \otimes \varphi(e, c)]$$
$$\leq \bigvee_{[e]_\theta \in \mathrm{G}/\theta} [\bigvee_{e' \in \mathrm{G}} (*\,(a, b, e') \otimes \varphi(e', c))]$$
$$= \bigvee_{[e]_\theta \in \mathrm{G}/\theta} \left[*^{\mathbf{G}/\varphi} ([a]_\varphi, [b]_\varphi, [c]_\varphi)\right]$$

$$=*^{\mathbf{G}/\varphi}\left([a]_\varphi,[b]_\varphi,[c]_\varphi\right).$$

Thus, h is a weak isomorphism.

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