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Estimations in Step-Partially Accelerated Life Tests for an Exponential Lifetime Model Under Progressive Type-I Censoring and General Entropy Loss Function

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Abstract

Based on progressively type-I censored samples, this paper discusses some estimation methods in step-partially accelerated life tests when the lifetimes of items under use condition follow the exponential distribution. Maximum likelihood estimations for the considered parameters are obtained in closed forms. The observed Fisher information matrix is derived to calculate confidence intervals for the considered parameters. Bayesian estimations for the parameters are carried out based on (a) informative prior for the scale parameter and discrete prior for the acceleration factor, (b) both the symmetric loss (squared error loss) function and asymmetric loss (general entropy loss) function. The resulting Bayes estimates are obtained in closed forms. The precision of the estimates and a comparison among them are investigated through a Monte Carlo simulation study.

Keywords: Partially accelerated life tests, Progressive type-I censoring, Exponential distribution, Maximum likelihood and Bayesian estimations, general entropy loss, Simulation.

1. Introduction

Censoring is of great importance in planning duration experiments in reliability and lifetime studies whenever the experimenter does not observe the lifetimes of all test units.

The traditional censoring schemes (type-I and type-II censoring) do not allow for units to be removed from the test at points other than the terminal point of the experiment. This allowance will be important when a compromise between reduced time of experimentation and the observations of at least some extreme lifetimes are sought. Also when some of the surviving units in the experiment that are removed early one can be used for some other test. These reasons lead us into the area of progressive censoring.

Accelerated life tests (ALTs) are often used for reliability analysis. Test units are run at higher-than-usual stress levels to induce early failures. A model relating life length to stress is fitted to the accelerated failure times and then extrapolated to estimate the failure time distribution under normal use condition.

The stress loading in an ALT can be applied in different ways: commonly used methods are *constant* stress, *progressive* stress and *step* stress. Nelson (1990, Chapter 1) discussed their advantages and disadvantages.

In ALTs the units are tested only at accelerated conditions (see AL-Hussaini and Abdel-Hamid (2006)) whereas in partially ALTs (PALTs) the units are tested at both accelerated and normal conditions. PALTs include two types, one is called step PALTs (see Abdel-Hamid and AL-Hussaini (2008)) and the other is called constant PALTs (see Abdel-Hamid (2009)).

The step PALT (which is considered in this paper) permits the test to be changed from normal use condition to accelerated condition at a predetermined time. Bai and Chung (1992) used the maximum likelihood (ML) method to estimate the scale parameter and the acceleration factor (the ratio of the main life at normal condition to that at accelerated condition) for exponentially distributed lifetime using type-I censoring data, and in (1993) Bai, et al obtained the same results when the lifetime is subjected to the log-normal distribution.

The novelty in this paper is to apply the step PALTs to the exponential distribution using progressively type-I censored data and then estimate the parameters under consideration using ML and Bayes methods.

Several authors investigated inferences under progressively censored data using different lifetime distributions, among others, Viveros and Balakrishnan (1994), Balakrishnan and Sandhu (1995, 1996), Balasooriya and Balakrishnan (2000), Ng, et al (2002, 2004) and Soliman (2005, 2008). Gouno, et al (2004) considered a k-level step-stress ALT under progressive type-I censoring while Wu, et al (2008) discussed the same problem considering progressive type-I censoring with grouped data. Balasooriya and Low (2004) discussed progressively type-I censored variable-sampling plans for Weibull lifetime distribution under competing causes of failures.

The exponential distribution is useful to describe failure times of units which subject to wear out. Pal, et al (2006, p.152) indicated that failure times of electric bulbs, appliances, batteries, transistors, etc can be modeled by exponential distribution. Therefore, this distribution is frequently discussed in reliability applications.

The rest of the paper is organized as follows: In Section 2, a description of the model and a discussion of progressive type-I censoring scheme are presented. Closed forms of the maximum likelihood estimates (MLEs) of the parameters under consideration are derived in Section 3. Based on the squared error loss (SEL) and general entropy loss (GEL) functions, Bayesian estimation of the parameters is obtained in Section 4. A simulation study and an illustrative example are presented in Section 5. Finally, some concluding remarks are given in Section 6.

2. Model Description and Progressive Type-I Censoring

According to step PALT each of n test units under consideration is first run at normal condition and if it does not fail by stress change time τ, then the test is changed to accelerated condition and held until all units fail. Suppose that Y is the total lifetime of a unit under normal and accelerated conditions. Thus

$$Y = \begin{cases} T, & 0 < T \le \tau, \\ \tau + (T - \tau)/\beta, & T > \tau, \end{cases} \tag{2.1}$$

where T is the lifetime of a unit at normal condition, τ is the stress change time and $\beta(>1)$ is the acceleration factor.

Suppose also that the random variable T has exponential distribution with scale parameter $\theta(>0)$. Thus the cumulative distribution function (CDF) of T is given by

$$F(t) = 1 - \exp(-t/\theta), \quad t > 0. \tag{2.2}$$

2.1 Progressive Type-I Censoring Scheme

The progressive type-I censoring is applied to step PALT as follows: The n test units are initially placed on normal stress condition and run until time $\tau_1(>0)$, at which point the number of failed units n_1 are counted and R_1 surviving units are removed from the test; starting from time τ_1 the remainder

$n-n_1-R_1$ surviving units are run until time τ_2 at which point the number of failed units n_2 are counted and R_2 surviving units are removed from the test. The test continues in this manner until time τ_k at which point the remainder $n-n_k-R_k$ surviving units are then placed on accelerated condition and run until time τ_{k+1} at which point the number of failures n_{k+1} are counted and R_{k+1} surviving units are removed from the test. The test continues in this manner under accelerated condition until time τ_K at which $R_K=n-\Sigma_{i=1}^{K}n_i-\Sigma_{i=1}^{K-1}R_i$ surviving units are removed, thereby terminate the test. The removed units are often used in other experiments in the same or at different facilities. The censoring times $\tau_1,\ldots,\tau_k,\ldots,\tau_K$ are fixed in advance.

The experimenter may notice the following iid observations:

$$\begin{aligned} &y_{11}\le\ldots\le y_{1n_1}\le\tau_1\le y_{21}\le\ldots y_{2n_2}\le\ldots\le y_{(k-1)1}\le\ldots\le y_{(k-1)n_{k-1}}\le\tau_{k-1}\le \\ &y_{k1}\le\ldots\le y_{kn_k}\le\tau_k\le y_{(k+1)1}\le\ldots y_{(k+1)n_{k+1}}\le\tau_{k+1}\le\ldots\le y_{K1}\le\ldots\le y_{Kn_k}\le\tau_K. \end{aligned} \tag{2.3}$$

The PDF of a unit under step PALT may be written as follows.

$$g(y)=\begin{cases} f_1(y)=\dfrac{1}{\theta}\exp\left(-\dfrac{y}{\theta}\right), 0<y\le\tau_k, \\ f_2(y)=\dfrac{\beta}{\theta}\exp\left(-\dfrac{\tau_k+\beta(y-\tau_k)}{\theta}\right), y>\tau_k. \end{cases} \tag{2.4}$$

The survival function (SF) and hazard rate function (HRF) of the random variable Y are given, respectively, by

$$S(y)=\begin{cases} S_1(y)=\exp(-y/\theta), \quad y\le\tau_k, \\ S_2(y)=\exp\{-[\tau_k+\beta(y-\tau_k)]/\theta\}, \quad y>\tau_k. \end{cases} \tag{2.5}$$

$$H(y)=\begin{cases} H_1(y)=1/\theta, \quad y\le\tau_k, \\ H_2(y)=\beta/\theta, \quad y>\tau_k. \end{cases} \tag{2.6}$$

3. Maximum Likelihood Estimation

Based on the censoring times ($\tau_1,\ldots,\tau_k,\ldots,\tau_K$) and progressively type-I censored sample given in (2.3), the likelihood function can be written in the form

$$L(\theta,\beta;\mathbf{y})\propto\prod_{i=1}^{k}\prod_{j=1}^{n_i}f_1(y_{ij})[1-F_1(\tau_i)]^{R_i}\prod_{i=k+1}^{K}\prod_{j=1}^{n_i}f_2(y_{ij})[1-F_2(\tau_i)]^{R_i}. \tag{3.1}$$

Based on Equations (2.4) and (3.1), the log-likelihood function $\ell(\theta,\beta;\mathbf{y})=\log L(\theta,\beta;\mathbf{y})$ is given by

$$\begin{aligned}\ell(\theta,\beta;\mathbf{y})\propto -N\log\theta-\frac{1}{\theta}\Bigg\{&\sum_{i=1}^{k}\left(n_iR_i\tau_i+\sum_{j=1}^{n_i}y_{ij}\right)+N_2\left(\tau_k-\theta\log\beta\right)\\&+\sum_{i=k+1}^{K}\left(n_iR_i(\tau_k+\beta(\tau_i-\tau_k))+\beta\sum_{j=1}^{n_i}(y_{ij}-\tau_k)\right)\Bigg\},\end{aligned} \tag{3.2}$$

where y_{ij} is the j^{th} observed failure in the i^{th} pre-specified time interval $]\tau_{i-1},\tau_i],\ \tau_0=0$, $N_1=\sum_{i=1}^{k}n_i\ (N_2=\sum_{i=k+1}^{K}n_i)$ is the total number of observed failures before(after) τ_k and $N=N_1+N_2$.

The likelihood equations are then given by

$$\frac{\partial\ell}{\partial\theta}=0=N_2\tau_k-N\theta+\sum_{i=1}^{k}\left(n_iR_i\tau_i+\sum_{j=1}^{n_i}y_{ij}\right)+\tau_k\sum_{i=k+1}^{K}n_iR_i+\beta(Q+W), \tag{3.3}$$

$$\frac{\partial\ell}{\partial\beta}=0=\frac{N_2}{\beta}-\frac{Q+W}{\theta}, \tag{3.4}$$

where $Q=\sum_{i=k+1}^{K}n_iR_i(\tau_i-\tau_k)$ and $W=\sum_{i=k+1}^{K}\sum_{j=1}^{n_i}(y_{ij}-\tau_k)$.

The ML estimates (MLEs) $\hat{\theta}$ and $\hat{\beta}$ of θ and β can be obtained by solving Equations (3.3) and (3.4) with respect to θ and β. Thus

$$\left.\begin{aligned} \hat{\theta} &= \frac{1}{N_1}\left(\tau_k \sum_{i=k+1}^{K} n_i(R_i+1) + \sum_{i=1}^{k} n_i R_i \tau_i + \sum_{i=k+1}^{K}\sum_{j=1}^{n_i} y_{ij}\right), \\ \hat{\beta} &= \frac{N_2\hat{\theta}}{Q+W}, \end{aligned}\right\} \tag{3.5}$$

where Q and W are as given above.

3.1 Approximate Confidence Interval

The local Fisher information matrix, $\mathbf{F}$, for MLEs $(\hat{\theta},\hat{\beta})$ is the 2×2 symmetric matrix of negative second partial derivatives of $\ell(\theta,\beta)$ with respect to θ and β. The inverse of $\mathbf{F}$ is the local estimate of the asymptotic variance-covariance matrix of $\hat{\theta}$ and $\hat{\beta}$, that is

$$\mathbf{F}^{-1} = \begin{pmatrix} \dfrac{-\partial^2\hat{\ell}}{\partial\theta^2} & \dfrac{-\partial^2\hat{\ell}}{\partial\theta\partial\beta} \\ \dfrac{-\partial^2\hat{\ell}}{\partial\beta\partial\theta} & \dfrac{-\partial^2\hat{\ell}}{\partial\beta^2} \end{pmatrix}^{-1} = \begin{pmatrix} \mathrm{var}(\hat{\theta}) & \mathrm{cov}(\hat{\theta},\hat{\beta}) \\ \mathrm{cov}(\hat{\theta},\hat{\beta}) & \mathrm{var}(\hat{\beta}) \end{pmatrix}, \tag{3.6}$$

where the caret ^ indicates that the derivative is calculated at $(\hat{\theta},\hat{\beta})$. The elements of this matrix can be easily obtained.

Following the general asymptotic theory of MLEs, the sampling distribution of $\dfrac{\hat{\theta}-\theta}{\sqrt{\mathrm{var}(\hat{\theta})}}$ and $\dfrac{\hat{\beta}-\beta}{\sqrt{\mathrm{var}(\hat{\beta})}}$ can be approximated by astandard normal distribution which is useful in constructing confidence intervals (CIs) for the unknown parameters.

A two sided $100(1-\alpha)\%$ normal approximation CIs for the two parameters θ and β can then be constructed as

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\theta})} \qquad \text{and} \qquad \hat{\beta} \pm z_{\alpha/2}\sqrt{\mathrm{var}(\hat{\beta})},$$

where $z_{\alpha/2}$ is the value of standard normal random variable leaving an area $\alpha/2$ to the right and both of $\sqrt{\mathrm{var}(\hat{\theta})}$ and $\sqrt{\mathrm{var}(\hat{\beta})}$ can be obtained from (3.6). The value $z_{\alpha/2}$ should be replaced by $t^{\star}_{\alpha/2}$ (the value of t-distribution leaving an area $\alpha/2$ to the right) if $n < 30$.

4. Bayesian Estimation

The Bayesian approach plays an important rule in analyzing failure data and other time-to-event. It has been proposed as an alternative procedure to traditional statistical perspective.

Due to the complicated computations arising from a general Bayesian procedure (see, for example, Abdel-Hamid (2008)), it is preferred to consider an alternative procedure which may be regarded as an approximation to a more general procedure. In this paper, we suppose that β is restricted to a finite number of values $\beta_1,\ldots,\beta_q$ with respective prior probabilities $p_1,\ldots,p_q$ such that $\sum_{a=1}^{q} p_a = 1$, i.e. $P(\beta=\beta_a)=p_a$, see Soliman (2008). The use of discrete distribution with equal probabilities for the scale parameter β resulted in a closed form expression for the posterior distribution. This cased an element of uncertainly, which is sometimes desirable in some cases. Furthermore, suppose that, conditional upon $\beta=\beta_a$, θ has inverted gamma (c_a, d_a) with density

$$\pi(\theta \mid \beta=\beta_a) = \frac{d_a^{c_a}}{\Gamma(c_a)}\theta^{-c_a-1}\exp\left[-\frac{d_a}{\theta}\right], \qquad \theta>0, (c_a, d_a > 0), \tag{4.1}$$

where c_a and d_a are chosen so as to reflect prior beliefs on θ given that $\beta=\beta_a$.

Based on (2.4), (3.1) and (4.1), our actual opinion about θ is summarized by the conditional posterior distribution of θ given $\beta=\beta_a$ which is given by Bayes theorem as

$$\pi^{\star}(\theta \mid \beta=\beta_a;\mathbf{y}) = \frac{\theta^{-(N+c_a+1)}\Psi_a^{N+c_a}\exp\left[-\Psi_a/\theta\right]}{\Gamma(N+c_a)}, \tag{4.2}$$

where

$$\left.\begin{aligned}\Psi_a &= \phi+\psi_a+d_a,\\ \phi &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}(y_{ij}+R_i\tau_i),\\ \psi_a &= \sum_{i=k+1}^{K}\sum_{j=1}^{n_i}\left[\tau_k+\beta_a(y_{ij}-\tau_k)+R_i(\tau_k+\beta_a(\tau_i-\tau_k))\right]\end{aligned}\right\} \tag{4.3}$$

On applying the discrete version of Bayes theorem, the marginal posterior probability distribution of β is

$$\begin{aligned}\pi_a = P(\beta=\beta_a\,|\,\mathbf{y}) &= \frac{\int_0^\infty p_a\pi(\theta\,|\,\beta=\beta_a)L(\theta,\beta)d\theta}{\int_0^\infty \sum_{a=1}^{q} p_a\pi(\theta\,|\,\beta=\beta_a)L(\theta,\beta)d\theta}\\ &= \frac{\beta_a^{N_2}p_a d_a^{c_a}\Gamma(N+c_a)\Psi_a^{-N-c_a}}{\Gamma(c_a)\Sigma_{a=1}^{q}\beta_a^{N_2}p_a(d_a^{c_a}/\Gamma(c_a))\Gamma(N+c_a)\Psi_a^{-N-c_a}},\end{aligned} \tag{4.4}$$

where Ψ_a is as defined in (4.3).

4.1 Estimation Based on Squared Error Loss Function

Under squared error loss (SEL) function, the Bayes estimator for θ is

$$\begin{aligned}\tilde{\theta}_{SE} &= \int_0^\infty \sum_{a=1}^{q}\theta\pi_a\pi^\star(\theta\,|\,\beta=\beta_a)d\theta\\ &= \sum_{a=1}^{q}\frac{\Psi_a}{N+c_a-1}\pi_a,\end{aligned} \tag{4.5}$$

and the Bayes estimator for β is given by

$$\tilde{\beta}_{SE} = \sum_{a=1}^{q}\beta_a\pi_a. \tag{4.6}$$

Similarly, the Bayes estimators for the SF and HRF at some $y=y^\star$ under SEL are given, respectively, by

$$\tilde{S}_{SE}(y^\star) = \sum_{a=1}^{q} \left[\frac{\Psi_a}{\Psi_a + (y^\star)^{\varepsilon} (\tau_k + \beta_a (y^\star - \tau_k))^{1-\varepsilon}} \right]^{N+c_1} \pi_a, \tag{4.7}$$

$$\tilde{H}_{SE}(y^\star) = \sum_{a=1}^{q} \frac{N + c_a}{\Psi_a} \beta_a^{1-\varepsilon} \pi_a, \tag{4.8}$$

where

$$\varepsilon = \begin{cases} 1, & y^\star \le \tau_k, \\ 0, & \tau_k < y^\star < \tau_K. \end{cases} \tag{4.9}$$

4.2 Asymmetric Loss Function

The loss function $L(\tilde{\vartheta}, \vartheta)$ provides a measure of the financial consequences arising from a wrong estimate $\tilde{\vartheta}$ of the unknown quantity ϑ. Due to its good mathematical properties, not its applicability to represent a true loss structure, most of the Bayesian inference procedures have been employed using symmetric SEL function (Tribus and Szonyi (1989) and Leon, et al (1992)),

$$L(\tilde{\vartheta}, \vartheta) = \zeta (\tilde{\vartheta} - \vartheta)^2, \tag{4.10}$$

where ζ is constant.

Although the quadratic loss function in (4.10) is a reasonable choice for many estimation problems, there are several situations where it is not appropriate. For example, during the estimation of the average reliable working life of the components of a space shuttle or an aircraft, over-estimation is usually more serious than under-estimation of the same magnitude, Kamińska and Porosiński (2009). So that a loss function should represent the consequences of different errors which may arise from over- and under-estimations. To overcome this problem, asymmetric loss functions have been introduced such as linear-exponential (LINEX) loss function, introduced by Varian (1975), and general entropy loss function (GEL), introduced by Calabria and Pulcini (1996). Despite the popularity and flexibility of the LINEX loss function to deal with estimation of the location parameter, it seems to be not appropriate for estimation of the scale parameter and other quantities, Basu and Ebrahimi (1991), Parsian and Sanjari Faripour (1993) and Srivastava and Tanna (2007).

A suitable alternative to LINEX loss is the GEL given by:

$$L(\tilde{\vartheta},\vartheta) \propto \left(\frac{\tilde{\vartheta}}{\vartheta}\right)^{\nu} - \nu \log\left(\frac{\tilde{\vartheta}}{\vartheta}\right) - 1, \tag{4.11}$$

whose minimum occurs at $\tilde{\vartheta} = \vartheta$.

This loss is a generalization of the entropy loss proposed by several authors where $\nu = 1$, Dey, et al (1987) and Dey and Liu (1992). For positive values of ν, a positive error ($\tilde{\vartheta} - \vartheta > 0$) causes more serious error than a negative error. The Bayesian estimator $\tilde{\vartheta}_{GE}$ of ϑ under GEL is given by

$$\tilde{\vartheta}_{GE} = \left(E_{\vartheta}[\vartheta^{-\nu}]\right)^{-1/\nu}, \tag{4.12}$$

provided existence and finiteness of $E_{\vartheta}[\vartheta^{-\nu}]$. It is clear that when $\nu = -1$, Bayesian estimator (4.12) coincides with the Bayesian estimator under the SEL function, whereas when $\nu = 1$ this estimator coincides with the Bayesian estimator of the SF under the weighted SEL function, $L(\tilde{\vartheta},\vartheta) = (\tilde{\vartheta} - \vartheta)^2 / \vartheta$.

4.3 Estimation Based on General Entropy Loss Function

It is shown above that under GEL (4.11) the Bayesian estimation of the unknown parameter/function can be calculated from Equation (4.12). Therefore, if in Equation (4.12) $\vartheta = \theta$, then the Bayesian estimator of the scale parameter θ of Equation (2.4) under GEL is given by

$$\begin{aligned}\tilde{\theta}_{GE} &= E\left[\theta^{-\nu}\right]^{-1/\nu} \\ &= \left[\int_0^{\infty} \theta^{-\nu} \sum_{a=1}^{q} \pi_a \, \pi^{\star}(\theta \mid \beta = \beta_a) d\theta\right]^{-1/\nu}.\end{aligned} \tag{4.13}$$

Using Equations (4.2) and (4.4), Equation (4.13) becomes

$$\tilde{\theta}_{GE} = \left[\sum_{a=1}^{q} \frac{\Gamma(N + c_a + \nu)}{\Gamma(N + c_a)\Psi_a^{\nu}} \pi_a\right]^{-1/\nu}. \tag{4.14}$$

Similarly put $\vartheta = \beta$, then the Bayesian estimator of the acceleration factor β under GEL is given by

$$\tilde{\beta}_{GE} = \left[\sum_{a=1}^{q} \beta_a^{-\nu} \pi_a \right]^{-1/\nu}. \tag{4.15}$$

Similarly, the Bayesian estimator of the SF and HRF at some $y = y^\star$ under GEL are given, respectively, by

$$\tilde{S}_{GE}(y^\star) = \left[\sum_{a=1}^{q} \left(\frac{\Psi_a}{\Psi_a - \nu (y^\star)^{\varepsilon} (\tau_k + \beta_a (y^\star - \tau_k))^{1-\varepsilon}} \right)^{N+c_a} \pi_a \right]^{-1/\nu}, \tag{4.16}$$

$$\tilde{H}_{GE}(y^\star) = \left[\sum_{a=1}^{q} \frac{\Gamma(N + c_a - \nu)}{\Gamma(N + c_a)} \left(\frac{\Psi_a}{\beta_a^{1-\varepsilon}} \right)^{\nu} \pi_a \right]^{-1/\nu}, \tag{4.17}$$

where ε is as defined in (4.9). The value of $y^\star$ has been taken to equal 0.3 in the simulation study.

5. Simulation Study

Due to the complicated expressions of the estimators, an analytical comparison of these estimators is impossible. Therefore, a Monte Carlo simulation study is carried out in order to calculate the MLEs, Bayes estimates (BEs), mean squared errors (MSEs), relative absolute biases (RABs) and 90% approximate CIs of the model parameters, based on $r = 1000$ Monte Carlo simulations.

The simulation study is performed according to the following steps

1. Generate a random sample of size n from uniform $(0,1)$ distribution and obtain the order statistics $(U_{1:n}, \ldots, U_{n:n})$.

2. For a given value of the parameter θ and a value of stress change time τ_k, find $n_1^\star$ such that

$$U_{n_1^\star:n} \le 1-\exp\left(-\frac{\tau_k}{\theta}\right) < U_{n_1^\star+1:n}.$$

3. For given values of acceleration factor β and censoring time τ_K, find $n_2^\star$ such that

$$U_{n_2^\star:n-n_1^\star} \le 1-\exp\left(-\frac{\tau_k+\beta(\tau_K-\tau_k)}{\theta}\right) < U_{n_2^\star+1:n-n_1^\star}.$$

4. From steps 2 and 3, the ordered observations

$$y_{1:n}^\star < \ldots < y_{n_1^\star:n}^\star \le \tau_k < y_{n_1^\star+1:n}^\star < \ldots < y_{n_1^\star+n_2^\star:n}^\star \le \tau_K,$$

are calculated as follows

$$y_{i:n}^\star = \begin{cases} -\theta\log(1-U_{i:n}), & 1 \le i \le n_1^\star, \\ \tau_k-(\theta\log(1-U_{i:n})+\tau_k)/\beta, & n_1^\star+1 \le i \le n_1^\star+n_2^\star. \end{cases}$$

5. The observations $y_{i:n}^\star, i=1,\ldots,n_1^\star+n_2^\star$ represent a type-I censored sample generated from the exponential distribution under PALT.

6. For given values of k and K, apply the progressive type-I censoring scheme to the observations generated in step 4 to obtain the observations given in (2.3), where $n_1^\star=\sum_{i=1}^{k} n_i+R_i$ and $n_2^\star=\sum_{i=k+1}^{K} n_i+R_i$.

7. Based on the progressively type-I censored sample given is step 5, calculate $\hat{\theta}$ and $\hat{\beta}$ according to Equations (3.5).

The MSE$(\hat{\theta})$ and RAB$(\hat{\theta})$ based on r different samples are calculated as follows

$$\text{MSE}(\hat{\theta}) = \frac{1}{r}\sum_{\omega=1}^{r}(\hat{\theta}_\omega-\theta)^2 \quad \text{and} \quad \text{RAB}(\hat{\theta}) = \frac{1}{r\theta}\sum_{\omega=1}^{r}|\hat{\theta}_\omega-\theta|.$$

8. Similarly, the MSE$(\hat{\beta})$ and RAB$(\hat{\beta})$ can be calculated as in step 7.

9. The BEs under SEL (GEL) of θ, β, SF and HRF with their MSEs and RABs can be computed similarly from Equations (4.5)-(4.8) ((4.14)-(4.17)).

Suppose that the progressive censoring is designed with three censoring times. The first two of them are occurred at use stress condition and the third one is occurred at accelerated stress condition. At the second censoring time, the stress is changed from use to accelerated condition. The experiment terminates at the time point which corresponds to the third pre-specified censoring time.

It has been taken into account that the calculations are performed on type-I censoring, $R_1 = R_2 = 0$, and progressive type-I censoring, $R_1 \neq 0, R_2 \neq 0$, for the sake of comparison.

Based on samples of sizes 25, 50 and 100 subject to progressive type-I censoring with two different censoring schemes (C.S.), Table 1 shows the MLEs with their MSEs and RABs. It shows also the lower and upper bounds of 90% CIs for the unknown parameters in addition to their lengths and coverage probabilities (COVPs). Table 2 shows the BEs with their MSEs and RABs of the model parameters in addition to the SF and HRF based on SEL and GEL functions with $\nu = -3,1,3$. The MLEs and BEs shown in Tables 1 and 2 are the average estimates over 1000 different samples.

The population parameter values used in the simulation study are $\theta = 0.65$ and $\beta = 1.2$. The censoring time values are $\tau_1 = 0.1$, $\tau_2 = 0.35$ and τ_3=6.0. The parameter β has assigned discrete distribution with ten values 1.05(0.02)1.23 with equal probabilities $p_a = 0.1,\ a = 1,\ldots,10$.

The MLE of the SF and HRF at some $y^\star > 0$ can be computed by using the invariance property of MLEs.

The following two points have been taken into account in the simulation procedure:

- The IMSL subroutines for pseudo-random number generation have been used.
- It has been numerically shown that the vector of parameters in the considered population satisfying the log-likelihood Equations (3.3)-(3.4) actually maximizes log-likelihood function (3.2). This is done by applying Theorem (7-9) on p. 152 of Apostol (1960).

Table 1. MLEs (CIs) of θ and β with their MSEs and RABs (lengths and estimated coverage probabilities (in %)) for different sample sizes and censoring schemes.

	C.S.	$\hat{\theta}$ $\hat{\beta}$	MSE($\hat{\theta}$) MSE($\hat{\beta}$)	RAB($\hat{\theta}$) RAB($\hat{\beta}$)	CI(θ) CI(β)	LCI(θ) LCI(β)	COVP(θ) COVP(β)
25	R_1=R_2=1	0.9730 1.8972	0.2074 1.0800	0.2926 0.3416	(-0.7706,2.1679) (-1.8218,4.5026)	2.9385 6.3245	79.6 82.4
	R_1=R_2=0	0.6986 1.3404	0.0592 0.3661	0.2447 0.3320	(-0.3879,2.3340) (-1.0614,4.8559)	2.7219 5.9173	91.6 83.1
50	R_1=R_2=1	0.9551 1.7493	0.1263 0.4973	0.2980 0.2920	(-0.5128,1.9007) (-0.9396,3.4295)	2.4136 4.3691	80.0 84.6
	R_1=R_2=0	0.6939 1.2449	0.0350 0.1575	0.1788 0.2559	(0.1349,1.7754) (0.0147,3.4840)	1.6405 3.4693	93.5 83.3
100	R_1=R_2=1	0.9267 1.6920	0.0892 0.3321	0.2889 0.2726	(0.1463,1.1969) (0.2901,2.1797)	1.0505 1.8896	86.9 85.6
	R_1=R_2=0	0.6716 1.2349	0.0126 0.0696	0.1225 0.1741	(0.5893,1.2641) (0.9684,2.4155)	0.6748 1.4471	96.4 89.4

C.S. ≡ Censoring Schemes & LCI ≡ length of CI.

Table 2. BEs of θ, β, SF and HRF with their MSEs and RABs under SEL and GEL functions for different sample sizes and censoring schemes.

n	C.S.	E.M.	$\tilde{\theta}$ / $\tilde{\beta}$	MSE($\tilde{\theta}$) / MSE($\tilde{\beta}$)	RAB($\tilde{\theta}$) / RAB($\tilde{\beta}$)	$\tilde{S}$ / $\tilde{H}$	MSE($\tilde{S}$) / MSE($\tilde{H}$)	RAB($\tilde{S}$) / RAB($\tilde{H}$)
25	R_1=R_2=1	Bayes(SE)	0.4702	0.0373	0.4149	0.5183	0.0150	0.2284
			1.1650	0.0012	0.0300	2.2227	0.5772	0.2939
		GE ($\nu=-3$)	0.5252	0.0200	0.2613	0.5536	0.0094	0.1455
			1.1659	0.0012	0.0292	2.0560	0.3391	0.2416
		GE ($\nu=1$)	0.4599	0.0209	0.4461	0.5133	0.0102	0.2409
			1.1625	0.0014	0.0322	2.1740	0.3088	0.2085
		GE ($\nu=3$)	0.4901	0.0292	0.3473	0.5391	0.0100	0.2768
			1.1583	0.0017	0.0359	1.9493	0.2707	0.2023
	R_1=R_2=0	Bayes(SE)	0.6104	0.0075	0.1232	0.6018	0.0022	0.0660
			1.1638	0.0013	0.0311	1.7114	0.0737	0.1224
		GE ($\nu=-3$)	0.6773	0.0085	0.1038	0.6295	0.0019	0.0466
			1.1648	0.0012	0.0301	1.6085	0.0753	0.1107
		GE ($\nu=1$)	0.5934	0.0089	0.1129	0.5973	0.0025	0.0722
			1.1612	0.0015	0.0333	1.6633	0.0466	0.1097
		GE ($\nu=3$)	0.6309	0.0067	0.1104	0.6197	0.0035	0.0723
			1.1566	0.0019	0.0374	1.5034	0.0411	0.1122
50	R_1=R_2=1	Bayes(SE)	0.4960	0.0267	0.3270	0.5411	0.0091	0.1702
			1.1824	0.0003	0.0149	2.0600	0.3189	0.2465
		GE ($\nu=-3$)	0.5004	0.0249	0.3129	0.5430	0.0086	0.1652
			1.1843	0.0002	0.0132	2.0815	0.3375	0.2541
		GE ($\nu=1$)	0.4909	0.0282	0.3405	0.5391	0.0095	0.1748
			1.1805	0.0004	0.0165	2.0391	0.2970	0.2388
		GE ($\nu=3$)	0.4850	0.0293	0.3526	0.5368	0.0097	0.1785
			1.1788	0.0005	0.0180	2.0180	0.2690	0.2305

	R_1=R_2=0	Bayes(SE)	0.5909 1.1850	0.0066 0.0002	0.1239 0.0126	0.5974 1.7260	0.0019 0.0618	0.0629 0.1126
		GE ($\nu=-3$)	0.5984 1.1871	0.0061 0.0002	0.1151 0.0109	0.5994 1.7429	0.0018 0.0698	0.0602 0.1189
		GE ($\nu=1$)	0.5845 1.1832	0.0073 0.0003	0.1329 0.0141	0.5956 1.7073	0.0020 0.0548	0.0654 0.1060
		GE ($\nu=3$)	0.5778 1.1813	0.0084 0.0003	0.1457 0.0158	0.5935 1.6914	0.0023 0.0324	0.0698 0.1027
100	R_1=R_2=1	Bayes(SE)	0.4252 1.2101	0.0515 0.0001	0.5368 0.0083	0.4912 2.3790	0.0089 0.2319	0.1664 0.2010
		GE ($\nu=-3$)	0.4299 1.2106	0.0493 0.0001	0.5189 0.0088	0.4944 2.3838	0.0079 0.2376	0.1575 0.2424
		GE ($\nu=1$)	0.4224 1.2095	0.0528 0.0001	0.5471 0.0078	0.4895 2.3631	0.0085 0.1951	0.1608 0.2167
		GE ($\nu=3$)	0.4205 1.2087	0.0537 0.0001	0.5544 0.0073	0.4886 2.3419	0.0087 0.1692	0.1633 0.2113
	R_1=R_2=0	Bayes(SE)	0.5124 0.1.2122	0.0196 0.0001	0.2719 0.0100	0.5546 1.9704	0.0017 0.0570	0.0525 0.1118
		GE ($\nu=-3$)	0.5173 1.2130	0.0186 0.0001	0.2616 0.0107	0.5566 1.9800	0.0017 0.0547	0.0539 0.0918
		GE ($\nu=1$)	0.5088 1.2116	0.0206 0.0001	0.2807 0.0096	0.5533 1.9568	0.0019 0.0354	0.0632 0.0095
		GE ($\nu=3$)	0.5053 1.2110	0.0217 0.0001	0.2906 0.0091	0.5518 1.9440	0.0021 0.0223	0.0596 0.0091

C.S. ≡ Censoring Schemes. & E.M. ≡ Estimation Method.

6. Conclusion

Censoring is common phenomenon in many life and fatigue tests. It has been shown by Viveros and Balakrishnan (1994) that the inference is practical when the sample data are subjected to a progressively censored experimental scheme. It has been discussed in this paper some results on statistical inference when the data are gathered according to step PALTs and collected under progressive type-I censoring scheme. We have obtained MLEs and BEs, in closed forms, for the two unknown parameters as well as the SF and HRF considering an exponential life model. The results are obtained under both symmetric and asymmetric loss functions. A simulation study has been conducted to examine the performance of the MLEs as well as the BEs under different sample sizes. From the simulation results, listed in Table 1, we observe the following:

1. For fixed values of n, the MSEs and RABs of the MLEs that correspond to $R_1 = R_2 = 1$ (progressive type-I censoring) are greater than those correspond to $R_1 = R_2 = 0$ (type-I censoring).
2. The LCI(θ) and LCI(β) (the COVP(θ) and COVP(β)) that correspond to $R_1 = R_2 = 1$ are greater (less) than those correspond to $R_1 = R_2 = 0$.
3. For fixed values of R_1 and R_2, the MSEs, RABs and LCIs (COVPs) decrease (increase) as n increases.

From the simulation results, listed in Table 2, we observe the following:

4. For fixed values of sample size n and censoring schemes, the MSEs and RABs of the BEs of θ, β and SF (HRF) increase (decrease) as ν increases.
5. For fixed values of n the MSEs and RABs of the estimates under progressive type-I censoring $R_1 = R_2 = 1$ are greater than those under type-I censoring $R_1 = R_2 = 0$.
6. For fixed values of censoring scheme, all the MSEs and RABs decrease as n increases.
7. Practically, the negative values in lower bounds of the CIs should be taken equal zero since the two parameters are positive.

It can be observed from Table 1 and Table 2 that the BEs are better than MLEs for small sample sizes and become the best for large sample sizes.

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