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A Synthetic Control Chart Reexpression Vector Variance for Process Multivariate Dispersion

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Abstract

A control charts Reexpression Vector Variance (RVV) can be used to monitoring the dispersion of multivariate process as an alternative to the control charts Generalized Variance (GV) is commonly used. The synthetic control chart RVV is built in a combination the standard control charts RVV (Shewhart class control chart) with the control chart of conforming run length (CRL). Average run length of the synthetic control charts VV be compared with the standard control charts VV, the standard control charts GV and the synthetic control charts GV. The result, the synthetic control chart RVV superior than standard GV chart, synthetic GV chart, and standard RVV chart to all of the condition change of the covariance matrix.

Keywords: Multivariate dispersion, vector variance, conforming run length, average run length.

1. Introduction

In the field of manufacturing industry, monitoring in a process of becoming an inevitability. This activity is done to continuous quality improvement. There are two phase in monitoring the process. Phase I consisting of the use of a control chart for (i) stage 1 ‘start-up stage’ in retrospective testing what 's the process in control when subgrup-sugrup first drawn; and (ii) stage 2, ‘future control stage’ which is testing whether the process remain in control when subgrup-subgrup future taken. In the multivariate characteristic, standard value with regard to the mean vector μ_0 and a covariance matrix Σ_0 (Alt and Bedewi, 1986).

The focus of this paper, development the control chart for monitoring multivariate dispersion. One such the measure of multivariate dispersion of that can be used is a reexpresion of vector variance (RVV). A RVV control chart built can be clasified in a class of a Shewhart control chart. In the previous researchers, a Shewhart control chart less sensitive in detecting small shifts of the process To improve its drawback Wu and Spedding (2000) have made a univariat control chart of the synthesis of cases is to control an average of the process. A control chart synthesis built by combining a Shewhart control chart and control chart conforming run length (CRL). The results show that the performance of a control chart synthetic average become more sensitive in detecting small shift. The idea of control chart synthesis, it has been adopted by some researchers as Huang and Chen (2005) to monitoring deviation standard, Ghute and Shirke (2008a) to controlling mean vector process and Ghute and Shirke (2008b) to monitoring the multivariate dispersion of the process with gernelaized variance (GV) statistics. In this paper the synthetic control chart to be built in to controling the multivariate dispersion process with statistics RVV.

To this intention, in this paper, first be be drawn about a control chart RVV. The next be drawn also a control chart CRL in general. As basic subjects of in this paper is developing an algorithm to synthetic control chart RVV for monitoring multivariate dispertion. For example cases will be gave at the end.

2. The Standard Control Chart RVV

2.1. Data

Suppose a multivariate process (p variables) with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ positive definit, in a notation of random vector $\boldsymbol{X}$, it is assumed $\boldsymbol{X} \sim N_p\left(\boldsymbol{\mu},\boldsymbol{\Sigma}\right)$. The multivariate dispersion process will be controlled by RVV defined by:

$$RVV = \left[Tr(S^2)\right]^{\frac{1}{2p}} \tag{1}$$

where, S is covariance with sample size n.

In control processes, $\boldsymbol{\mu} = \boldsymbol{\mu}_0$ and $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_0$

Based on the limiting distribution (1), control limits RVV for phase II (Suwanda, 2014) is:

$$\begin{aligned} UCL &= \upsilon_{\Sigma_0} + k\tau_{\Sigma_0} \\ Central &= \upsilon_{\Sigma_0} \\ LCL &= \upsilon_{\Sigma_0} - k\tau_{\Sigma_0} \end{aligned} \tag{2}$$

where,

$$\upsilon_{\Sigma_0} = \left[Tr(\Sigma_0^2) \right]^{1/(2p)} \tag{3}$$

$$\tau_{\Sigma_0}^2 = \left[\frac{1}{4p^2} \frac{\sigma_{\Sigma_0}^2}{\left[Tr(\Sigma_0^2) \right]^{(2p-1)/p}} \right] \tag{4}$$

$$\sigma_{\Sigma_0}^2 = \frac{8n}{(n-1)^2} Tr(\mathbf{\Sigma}_0^4) \tag{5}$$

and k = the constant to determine the probability that process out of control with the actual state process in control (α).

Can be shown (see the appendix) that the average run length in control (2) is

$$ARL_0 = 1/\alpha = 1/2\Phi(-k) \tag{6}$$

while that the average run length out of control:

$$ARL_1 = 1/P \tag{7}$$

where

$$P = 1 - \beta = 1 - \left[\Phi(b) - \Phi(a) \right]$$

$$a = \frac{\nu_{\Sigma_0} - k\tau_{\Sigma_0} - \nu_{\Sigma_1}}{\tau_{\Sigma 1}}$$

$$b = \frac{\nu_{\Sigma_0} + k\tau_{\Sigma_0} - \nu_{\Sigma_1}}{\tau_{\Sigma 1}}$$

3. The CRL Control Chart

The CRL control chart proposed by Bourke (1991) developed originally to monitoring the quality of attributes to detect shift in fractions defective, proportions when the inspection 100 % is used. The runs length of conforming items between to items nonconforming successive taken as a basis for a control chart.

The CRL defined as corresponding quantity of units is between the two nonconforming successive including the last unit is not conforming. The CRL can be explained in the next picture:

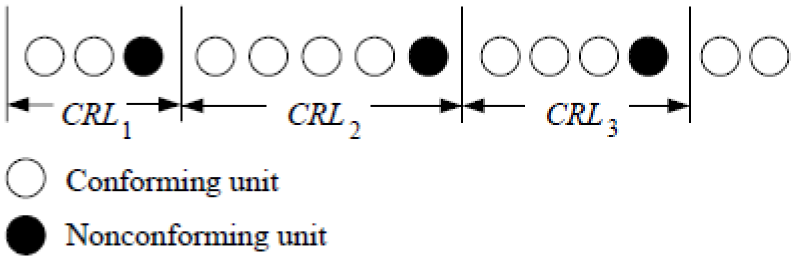

Figure 1. Calculate of CRL

Mean and the cumulative probability function of CRL are:

$$\mu_{CRL} = \frac{1}{\theta}$$

$$F_{\theta}(CRL) = 1 - (1-\theta)^{CRL}, \quad CRL = 1, 2, \ldots$$

In the case of detecting increasing of θ, the only of lower control limit used as follows:

$$LCL_{CRL} = L = \frac{\ln(1-\alpha_{CRL})}{\ln(1-\theta_0)} \tag{8}$$

where $\alpha_{CRL} = 1-(1-\theta_0)^L$ = error type I, θ_0 = proportion of nonconforming in control. L will be must integer.

If the sample CRL less than L, should be suspected that the proportion of damage has been increased.

The Control chart synthetic for RVV can be develoved by combining the control chart RVV with control chart CRL.

The Average Run Length (ARL) for the control chart CRL is:

$$ARL_{CRL} = \frac{1}{F_{\theta}(L)} = \frac{1}{1-(1-\theta)^L} \tag{9}$$

4. The Synthetic Control Chart RVV

This chart was created with the combining the control chart RVV and CRL. The mechanism the state of process as follows. If the RVV fall in the control limits, note as the state conforming with the symbol circle empty and if RVV falling in outside the control limits, note as the state of the nonconforming with a symbol of black circle (see Figure 1). In the synthetic control chart, L and k determined that filled ARL_0 (10) as expected (e.g. 370 to control chart 3 sigma) and the ARL_1 (11) minimum .In this ARL_0 for the synthetic control chart is the multiplication of (6) and (9), namely:

$$ARL_{0S} = ARL_{0RVV} \times ARL_{0CRL} = \frac{1}{2\Phi(-k)} \frac{1}{1-\left(1-2\Phi(-k)\right)^L} \tag{10}$$

and ARL_1 for Synthetic control chart is the multiplication (7) and (9), namely:

$$ARL_{1S} = ARL_{1RVV} \times ARL_{1CRL} = \frac{1}{P} \frac{1}{1-(1-P)^L} \tag{10}$$

The operation of synthetic control chart RVV are summarized as follows:

1. Set UCL and LCL from sub chart $RVV/\sin$ and LCL L from sub chart $CRL/\sin$ (determination of the control limits will be explained later).
2. On each inspection point, at i[th] period get a sample with size n observations $\boldsymbol{x}_i$, and calculate the sample matrix covariance $\boldsymbol{S}_i$ and RVV_i. This RVV_i sample plotted on a sub chart $RVV/\sin$.
3. If the RVV_i greater than LCL/sin and less than UCL/sin, this sample is called a appropriate sample (conforming) and monitoring returns to step 2. Otherwise sample referred to is not appropriate (nonconforming) and controlling continues to the next step.
4. Check the number of sample between sample now and the sample nonconforming. The number is extracted as a sample CRL from of sub-chart CRL/sin and the synthetic chart.
5. If the sample CRL is larger than the lower limit sub-diagram, the process in control and controlling next to step 2. Otherwise process it is out of control and controlling for the next step.
6. *Signal out of control* condition.

7. Make the act to obtain and remove the factor causes of out of control. And than summarize the process and then back to step 2.

To design a synthetic control chart, users should provide specifications follows:

The covariance matrix *in control* process, Σ_0.

Standard deviation of RVV, τ.

Sample sizel, n.

Covariance matrix out of control, Σ_1.

Average run length in control, ARL_0.

Usually, μ and Σ estimated from the data observations on the pilot runs and the sample size n not have to be large from the number of variables, p. Dispersion shift design is the distance through which it considered to be serious enough its impact on the quality of the product; then pertaining to a value ARL_1 that is as small as possible. The ARL_0 decided by the fulfillment of the level of any alarm. If handling error alarm is difficult, ARL_0 must large, other ARL_0 have to be set with a value of less to enhance the effectiveness of the detection.

A description of the program

This program can use to design parameters controlling $BKA_{RVV/Sin}$ and L that ensure the minimum for ARL to shift VV or change of covariance matrix for the synthetic described above. Data entry for flexibility, program determined by type error I, $\alpha\ (1/ARL_0)$.

The program design a synthetic chart based on the model optimization as follows:

The objective:

$$ARL_1 = \text{minimum} \tag{12}$$

Constrain

$$ARL_0 = \frac{1}{2\Phi(-k)} \times \frac{1}{1-\left[1-2\Phi(-k)\right]^L} \tag{13}$$

The two desigm variable:

L, k

where $ARL_1(\Sigma_1)$ = ARL *out of control.*

The optimization identify the set of L and k so $ARL_1(\Sigma_1)$ minimum with ARL_0 be fixed.

The procedures complete of design programs can be summarized as follows:

1. Set μ, Σ_0, n, Σ_1 and ARL_0 or α.
2. An initial value L as 1.
3. Determine *k* with solving (2) numerically, as follows:

   Let $ARL(0) = a$ and $\Phi(-k) = b$, Equation (2) into:

$$a = \frac{1}{2b} \times \frac{1}{1 - [1 - 2b]^L}$$

$$2b - [1 - 2b]^L 2b - \frac{1}{a} = 0$$

$$f(b) = 2b - [1 - 2b]^L 2b - \frac{1}{a} = 0$$

With Newton's method the roots of *b* can be determined with the following procedure:

Taylor series $f(b)$ at this point b_0 are:

$$f(b) = f(b_o) + \frac{f'(b_o)}{1!}(b - b_0) + \frac{f''(b_o)}{2!}(b - b_0)^2 + \frac{f'''(b_o)}{3!}(b - b_0)^3 + ... = 0$$

The linear approximation is:

$$f(b_o) + \frac{f'(b_o)}{1!}(b - b_0) \approx 0$$

$$(b - b_0) = -\frac{f(b_o)}{f'(b_o)}$$

$$b_1 = b_0 - \frac{f(b_o)}{f'(b_o)}$$

In this case $0 < \Phi(-k) = b < 0.5$, because the initial value can be between 0 and 0.5.

4. Calculate ARL_1 for the values k and L using the following equation

$$ARL_1 = \frac{1}{P} \times \frac{1}{1-(1-P)^L}$$

where P as at Equation 7.

5. If ARL_1 being reduced, then stepped up L by the addition of one and go back to step-3. Other to the next step.
6. Let L and k as final value for the shyntetic chart.
7. Use the final value of k, calculate $UCL_{RVV/Sin}$ and $LCL_{RVV/Sin}$ as follow :

$$LCL_{RVV/Sin} = \nu_0 - k\tau_0 , UCL_{RVV/Sin} = \nu_0 + k\tau_0$$

Ilustration:

To better understand the process of a synthetic chart RVV, this following will be given an example case. Suppose a process involving three important variables with a covariance matrix,

$$\Sigma_0 = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}$$

The monitoring done during the process of continuing with the use of the sample size n = 5. A syhthetic control chart RVV will be made to control to multivariate dispersion so sensitive to change all variable variance, increased to $\sigma_{ii} = 1.5, i = 1,2,3$ and the correlation between variables fixed equal to zero with $ARL_0 = 370$.

By using matlab R2008b obtained the values of L, k, LCL, UCL, and ARL_1 for the synthetic chart RVV, as articulated in the following table:

Table 1. The Values k, LCL, UCL and ARL_1 for L 1 to 20 and $ARL_0 = 370$

L	**k**	**LCL**	**UCL**	**ARL_1**
1	1.943	0.846	1.556	25.305
2	2.085	0.820	1.582	20.005
3	2.164	0.806	1.596	17.716
4	2.219	0.796	1.606	16.408
5	2.260	0.788	1.614	15.561
6	2.294	0.782	1.620	14.972
7	2.322	0.777	1.625	14.546
8	2.346	0.772	1.630	14.228
9	2.366	0.769	1.633	13.987
10	2.385	0.765	1.637	13.802
11	2.402	0.762	1.640	13.660
12	2.417	0.759	1.643	13.552
13	2.430	0.757	1.645	13.470
14	2.443	0.755	1.647	13.409
15	2.455	0.752	1.649	13.365
16	2.466	0.750	1.651	13.336
17	2.476	0.749	1.653	13.319
18	**2.486**	**0.747**	**1.655**	**13.312**
19	2.495	0.745	1.657	13.314
20	2.503	0.744	1.658	13.323

In the Table 1, for L = 18 obtained $ARL_1 = 13.312$, while in L = 19 give $ARL_1 = 13.314$. Hence the optimal ARL_1 for synthetic chart RVV happened in L = 18. So the limits of control RVV synthtetic:

Sub control chart RVV/Syn:

$$UCL_{RVV/Sin} = 1.655$$
$$LCL_{RVV/Sin} = 0.747$$

Sub control chart CRL/Sin:

$$LCL_{CRL/Sin} = L = 18$$

with ARL_1=13.314.

Whereas the limits of the control chart RVV standard are:

$$\begin{aligned} UCL_{RVV} &= 1.7491 \\ Central_{RVV} &= 1.2009 \\ LCL_{RVV} &= 0.5628 \end{aligned}$$

with ARL_1 = 27.0129.

It appears that in cases of the number of variables p = 3 and sample size n=5, a synthetic control chart RVV on average in the period of 13 or 14 will give the signal has happened changes RVV which was originally RVV_0 = 1.2009 be RVV_1 = 1.3747 or the ratio change PRVV = RVV_1 / RVV_0 = 1.14. The provision of this signal more quickly compared with an average length of the period of the provision of signals out of control the first time by a control chart RVV standard that is on the period 27 since the occurrence of a shift RVV to 27. Ghute and Shirke (2008) has calculated ARL1 for control chart GV standard ang synthetic control chat GV. To the case of p = 3, n = 5 and the ratio of change GV 1.2, standard give ARL1 = 75.56 fot control chart GV standar and give ARL1 = 56.84 for synthetic control chart GV. So a synthetic control chart RVV have the best performance to this case.

5. Performance Comparison with Other Control Chart

It is only will be presented the results in comparison to the case bivariate normal as has been done Ghute and Shirke (2008) .The results of other not shown in this paper.

Calculation ARL_1 done to sample size 4 with covariance matrix in control is a matrix identity. A covariance matrix out of control taken with provide change at one variance and or both of them.

A control chart synthetic RVV be compared with the control chart standard RVV, control chart standard GV$\left(|S|\right)$ and control chart synthetic GV. The results of comparison outlined in table the following.

Table 2. The Comparison Between Synthetic RVV Chart and GV Chart, Synthetic GV Chart and RVV Chart on $n = 4, p = 2$ and $\alpha = 0.005$.

$(\sigma_1, \sigma_2, \rho)$	$\frac{GV_1}{GV_0}$	$\frac{RVV_1}{RVV_0}$	**Standard GV**	**Synthetic GV**	**Standard RVV**	**Synthetic RVV**
$(1.00, 1.00, 0)$	1.0	1.0	200	200	200	200
$(1.10, 1.00, 0)$	1.210	1.205	110.65	88.59	177.26	80.98
$(1.10, 1.10, 0)$	1.464	1.210	65.04	43.87	119	51.2
$(1.25, 1.00, 0)$	1.563	1.281	54.89	35.32	44.85	18.64
$(1.5, 1.00, 0)$	2.250	1.625	23.79	12.93	9.46	4.77
$(1.25, 1.25, 0)$	2.441	1.560	20.2	10.85	26.82	11.11
$(1.50, 1.50, 0)$	5.063	2.25	6.32	3.65	6.25	3.26

In Table 2 shows clearly that synthetic RVV chart superior than standard GV chart, synthetic GV chart, and standard RVV chart to all of the condition change of the covariance matrix.

6. An application in The Real Problem

Data retrieved from PT. Dirgantara Indonesia (PT DI) consists of 31 products in the form of wing components with three key characteristics. This Data is considered to be data in Phase I. To specify data that is out of control will be determined using a statistical outlier detection Wilks (1963). The results show data no. 25 to the outer bounds of the control. After this data is not entangled and recalculated all the data is on the boundaries of the control. Estimates of mean vector and covariance matrix by using clean data each are:

$$\hat{\mu}^t = \begin{pmatrix} 0.5189 & 1.8008 & 0.0000 \end{pmatrix}$$

and

$$\hat{\Sigma} = \begin{pmatrix} 0.0127 & -0.0024 & 0.0035 \\ -0.0024 & 0.0121 & 0.0006 \\ 0.0035 & 0.0006 & 0.0042 \end{pmatrix}$$

For the control of Phase II, the view $\mu_0 = \hat{\mu}$ and $\Sigma_0 = \hat{\Sigma}$. Scenarios used are as follows: in the first 20 period data first raised from the multivariate normal distribution with mean vector μ_0 and covariansi matrix Σ_0 with sample size n=5 for each period. In period 21 to 40 raised from the same distribution with mean vector μ_0 and th first two variances are increased 100% that is $\sigma_{11}^2 = 2\sigma_{10}^2$ and $\sigma_{21}^2 = 2\sigma_{20}^2$, so that the matrix kovariansi be :

$$\Sigma_1 = \begin{pmatrix} 0.0254 & -0.0024 & 0.0035 \\ -0.0024 & 0.0242 & 0.0006 \\ 0.0035 & 0.0006 & 0.0042 \end{pmatrix}$$

From the results of the calculation with $\alpha = 0.0027$, the obtained control limits for standard RVV chart as follows:

$$\begin{aligned} UCL_{RVV} &= 0.421 \\ Central_{RVV} &= 0.267 \\ LCL_{RVV} &= 0.113 \end{aligned}$$

While the control limits of synthetic RVV chart is:

Sub control chart RVV:

$$\begin{aligned} UCL_{RVV/Sin} &= 0.391 \\ LCL_{RVV/Sin} &= 0.143 \end{aligned}$$

Sub control chart CRL/Sin:

$$LCL_{CRL/Sin} = L = 12$$

The values of RVV sample for the 40 periods set out in Table 3, and plotted on Figure 2.

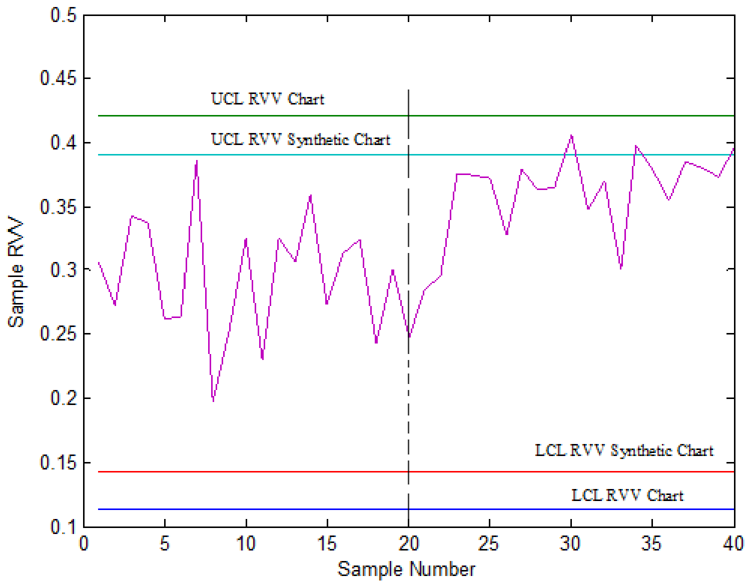

Figure 2. Standard RVV and Synthetic RVV Control Chart

Table 3. The result of Computed RVV for 40 Sample

No Smpl	RVV	No	RVV
1	0.306708	21	0.284688
2	0.272582	22	0.296055
3	0.342809	23	0.375864
4	0.337143	24	0.374261
5	0.261683	25	0.37157
6	0.264031	26	0.327729
7	0.38629	27	0.379001
8	0.198012	28	0.362808
9	0.253693	29	0.365216
10	0.324867	30	0.406345
11	0.230978	31	0.346926
12	0.325043	32	0.369146
13	0.306083	33	0.300865
14	0.359542	34	0.398055
15	0.274082	35	0.379293

16	0.313324	36	0.354092
17	0.324274	37	0.385115
18	0.242995	38	0.380633
19	0.300795	39	0.372806
20	0.247512	40	0.396432

It appears that at 20 the first sample not give a signal out of control since it was shifting from rvv not happen. On 21th period happened shift RVV from 0.267 to 0.3297. To the use of synthetic rvv control chart, in point 21 take CRL = 0. It appears that the period on the 30th, RVV = 0.406 great than $UCL_{RVV/syn} = 0.391$ with a value of CRL, L = 10 smaller than $LCL_{CRL/syn} = 12$. Hence on period 30 has happened out of control signals, because it has changed into RVV = 0.3297 since period 21th, while by standard RVV chart up to the period 40 this change has not been detected .

7. Conclusion

In this paper have been introduced a synthetic control chart to monitoring covariance matrix for the process distributed multivariate normal. A synthetic control chart RVV is a combination of control chart RVV and control chart CRL. In the case of number of variables p = 3 and sample size n = 5, the synthetic RVV chart superior than standard GV chart, synthetic GV chart, and standard RVV chart to all of the condition change of the covariance matrix. Implemtation at real problem for number of variable p = 3 and sample size n = 5 with scenario RVV changed from 0.267 start to 0.3297 at period 21th, the synthetic control chart RVV faster give the signal out of control that is in the period 10th. Whereas a standard RVV chart not gave signal. So a synthetic control chart RVV have the best performance to this case.

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Appendix

ARL for Control Chart RVV:

Average run length is defined as expectations of random variable geometric with the parameter is a probability of state out of control (the proportion of nonconforming), let P.

So,

$$ARL = \frac{1}{P}$$

Under the process of actually in control, $P = \alpha$, so

$$ARL_0 = \frac{1}{\alpha}$$

Under the process of actually out of control, $P = 1-\beta$, so

$$ARL_1 = \frac{1}{1-\beta}$$

Now will be determined value of β,

$$\begin{aligned}\beta &= P\left(BKB < RVV < BKA \middle| \Sigma = \Sigma_1, \Sigma_1 = \Sigma_0\right) \\ &= P\left(\frac{BKB - \nu_{\Sigma_1}}{\tau_{\Sigma_1}} < Z < \frac{BKA - \nu_{\Sigma_1}}{\tau_{\Sigma_1}}\right) \\ &= P\left(\frac{\nu_{\Sigma_0} - k\tau_{\Sigma_0} - \nu_{\Sigma_1}}{\tau_{\Sigma_1}} < Z < \frac{\nu_{\Sigma_0} + k\tau_{\Sigma_0} - \nu_{\Sigma_1}}{\tau_{\Sigma_1}}\right) \\ &= P\left(a < Z < b\right) = \Phi\left(b\right) - \Phi\left(a\right)\end{aligned}$$

where

$$a = \frac{\nu_{\Sigma_0} - \nu_{\Sigma_1} - k\tau_{\Sigma_0}}{\tau_{\Sigma_1}}$$

$$b = \frac{\nu_{\Sigma_0} - \nu_{\Sigma_1} + k\tau_{\Sigma_0}}{\tau_{\Sigma_1}}$$

Under process in control is true:

$$a = \frac{\nu_{\Sigma_0} - \nu_{\Sigma_0} - k\tau_{\Sigma_0}}{\tau_{\Sigma_0}} = -k$$

$$b = \frac{\nu_{\Sigma_0} - \nu_{\Sigma_0} + k\tau_{\Sigma_0}}{\tau_{\Sigma_0}} = k$$

thereby,

$$\beta = \Phi\left(k\right) - \Phi\left(-k\right)$$

And

$$\begin{aligned}ARL_0 &= \frac{1}{1 - \left[\Phi\left(k\right) - \Phi\left(-k\right)\right]} \\ &= \frac{1}{2\Phi\left(-k\right)} \\ &= \frac{1}{\alpha}\end{aligned}$$