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Constant Proportion Portfolio Insurance Strategies in Fuzzy Financial Markets

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Abstract

Portfolio insurance techniques have been in existence for a long period of time. Cont and Tankov (2007) propose that portfolio insurance refers to investment strategies that limit the downside risk of a portfolio whilst maintaining its upside potential at the same time. Constant proportion portfolio insurance (CPPI) is a fundamental and prominent example of portfolio insurance strategies. Several authors have analysed CPPI based on probability theory (for example, Neftci, 2008 and Cont and Tankov, 2007) and uncertainty theory (such as, Matenda, Chikodza and Gumbo, 2015). Probability theory and uncertainty theory recognise randomness and uncertainty, respectively, as the only legitimate forms of indeterminacy. However, there are other forms of material indeterminacy in financial markets, such as fuzziness, which cannot be modelled by probability theory and uncertainty theory. In order to deal with fuzziness, credibility theory was founded. The main aim of this research work is to analyse the mechanics of CPPI strategies in fuzzy financial markets. Assuming continuous time diffusion models, CPPI techniques always work. However, in practice, CPPI strategies are exposed to gap risk which originates from sudden significant downward asset price jumps. In this research paper a direct relationship between the participation rate, and the CPPI-insured portfolio value, has been established. The risk of loss in a CPPI strategy increases with the participation rate. Gap risk for CPPI strategies is not insignificant. Therefore, it has to be quantified. This research paper develops a strong foundation for the analytical computation of gap risk for CPPI strategies when asset price processes evolve as fuzzy differential equations with jumps. This study is the first peace of work to apply credibility theory to CPPI.

Keywords: Portfolio insurance, indeterminacy, fuzziness, credibility theory, gap risk, fuzzy markets, participation rate.

1. Introduction

Financial markets are characterised by a lot of indeterminate phenomena. Indeterminacy refers to phenomena whose outcomes cannot be precisely predicted in advance (Peng, 2013). Matenda, Chikodza and Gumbo (2015) suggest that indeterminacy is conceptualised as a state of events' outcomes being unpredictable in advance. Stock price, tossing a dice, tossing a coin, roulette wheel, life time, heavy loss and product demand are some of the examples of indeterminate phenomena. In order to model indeterminacy, several mathematical theories have been suggested. Probability theory (Kolmogorov, 1933), fuzzy set theory (Zadeh, 1965) and uncertainty theory (Liu, 2007) are the most basic and prominent examples of mathematical systems developed to deal with indeterminacy. Fuzzy set theory models fuzziness, uncertainty theory deals with uncertainty and probability theory recognises randomness.

Portfolio insurance refers to portfolio management strategies which promise that the portfolio value at any time up to maturity will not fall below a specified floor, usually gazetted as a percentage of the initial investment (Cont and Tankov, 2007). It is widely documented and understood that CPPI is one of the fundamental and prominent examples of portfolio insurance techniques. The notion of CPPI was first introduced by Perold (1986) for fixed income instruments and Black and Jones (1987) for equity instruments. Several authors have examined CPPI based on probability theory (for example, Neftci, 2008 and Cont and Tankov, 2007) and uncertainty theory (such as, Matenda, Chikodza and Gumbo, 2015). Probability theory is an axiomatic branch of pure mathematics that models the dynamics of random phenomena. The study of probability was initiated by Pascal and Fermat in 1654 and the foundation of probability theory was proposed by Kolmogorov (1933). Stochastic finance theory is based on probability theory. Uncertainty theory was pioneered by Liu (2007) and further revised by Liu (2010). The theory of uncertainty is defined as an axiomatic branch of pure mathematics for modelling human uncertainty. Uncertain finance theory has its roots in uncertainty theory.

Zadeh (1965) suggests the fuzzy set via membership function which can be proposed by experienced experts. Since Zadeh's 1965 work, several mathematical systems have been introduced to deal with human systems which include the theory of capacities (Choquet, 1954) and fuzzy measure (Sugeno, 1974). Among other issues, capacity and fuzzy measure are heavily criticised on the grounds that they do not obey the law of truth conservation and they are inconsistent with the law of contradiction and the law of excluded middle (Liu, 2012). Liu (2012) also laments that product capacity and product fuzzy measure have not been defined. Zadeh (1978) introduces a possibility measure and Nahmias (1978) suggests the axioms of a possibility measure in order to model fuzziness. A possibility measure has its share of criticisms which gave Liu and Liu (2002) the impetus to introduce a credibility theory to effectively deal with fuzziness. Liu (2012) propounds that a possibility measure does not obey the law of truth conservation and is inconsistent with the law of contradiction and the law of excluded middle.

A credibility measure was suggested by Liu and Liu (2002) in order to measure a fuzzy event. Li and Liu (2006) introduce a sufficient and necessary condition for a credibility measure. The theory of credibility was pioneered by Liu (2004) and further improved by Liu (2007). Credibility theory is an axiomatic branch of pure mathematics that models and analyses the behaviour of dynamic fuzzy phenomena which is based on the following axioms: normality, monotonicity, self-duality and maximality.

In order to model dynamic fuzzy phenomena which vary with time or space, Liu (2008) introduced a notion of a fuzzy process. In general, a fuzzy process is a sequence of fuzzy variables which vary with time. A Liu process and a renewal process are two fundamental and important types of fuzzy processes. Liu (2008) introduces a Liu process, which is defined as a stationary and independent increment fuzzy process whose increments are normally distributed fuzzy variables. Peng (2008) propounds that a Liu process and a geometric Liu process are regarded as the fuzzy counterparts of a Brownian motion and a geometric Brownian motion, respectively. A renewal process was first proposed by Zhao and Liu (2003) and is defined as a fuzzy process in which events occur continuously and independently of each other in fuzzy times.

Based on a Liu process, Liu (2008) proposes the notion of fuzzy calculus. Fuzzy calculus is a branch of pure mathematics that deals with the integration and differentiation of fuzzy processes. Liu (2008) introduces a Liu integral which is a fuzzy integral with respect to a Liu process. In order to differentiate functions of a Liu process, Liu (2008) introduces a Liu formula. The existence and uniqueness theorem for homogeneous fuzzy differential equations was provided by You (2008). A differential equation which is driven by a Liu process is called a fuzzy differential equation. Moreover, Liu (2008) suggests fuzzy calculus with jump processes in order to deal with jumps in fuzzy processes. A fuzzy differential equation with jumps is a differential equation which is driven by both a Liu process and a renewal process.

The application of fuzzy mathematics in finance gave birth to fuzzy finance theory. Fuzzy calculus and fuzzy differential equations were first introduced in the discipline of finance by Liu (2008) under the assumption that stock prices follow a geometric Liu process. Liu (2008) proposes a fuzzy stock model named a Liu's stock model which is regarded as a fuzzy counterpart of a Black-Scholes model. In order to incorporate asset price shocks into a stock model, Liu (2008) suggests a fuzzy stock model with jumps.

For more technical and detailed expositions on the application of fuzzy calculus in finance the reader is referred to, among other sources, Liu (2008), Qin and Li (2008), Gao (2008), Peng (2008), Yoshida et al. (2006) and Yoshida (2003).

CPPI

The main aim of CPPI techniques is to guarantee the initial principal amount of a market participant, F, at the end of the investment horizon, T (Matenda, Chikodza and Gumbo, 2015). Conceptually, CPPI limits the downside risk of a CPPI-insured portfolio whilst maintaining its upside potential. However, as compared to an unprotected portfolio, the upside potential of a CPPI-insured portfolio is reduced (Gerber and Pafumi, 2000).

A CPPI strategy is heavily premised on the cushion, U_t, which is defined as the difference between the total portfolio value, V_t, and the portfolio lower bound (known as the floor), P_t. That is, $U_t = V_t - P_t$. The portfolio floor is the present value of the initial principal investment, F, to be received at time T. A CPPI-insured portfolio value is not allowed to fall below the lower bound in order to guarantee that the market participant will receive at least the initial principal investment at time T. Portfolio value is taken to be the total market value of a risky asset and a default-free bond.

In a CPPI strategy, part of the portfolio is invested in a risky asset, for example, a financial index such as the FTSE All-Share or a portfolio of stocks and the remainder is placed in a riskless security, usually a default-free government bond. An amount to be placed in an underlying risky asset, Y_t, is proportional to the cushion. Mathematically, $Y_t = mU_t$, where m is known as the participation rate or multiplier. The participation rate represents the leverage level the investor is prepared to tolerate (Matenda, Chikodza and Gumbo, 2015). For a CPPI strategy to achieve its main aim of principal protection, m is assumed to be greater than or equal to one ($m \geq 1$) and constant throughout the investment tenure. In order to determine the amount to be invested in a zero-coupon bond, A_t, Y_t is deducted from V_t, that is, $A_t = V_t - Y_t$. In some cases, A_t is less than the time t value of a default-free bond which matures at T and whose future value is equivalent to F. In a CPPI strategy, the following conditions hold

(i). If $V_t > P_t$, Y_t is placed in a risky asset and A_t is invested in a riskless security.

(ii). If $P_t \geq V_t$, all money is invested in a default-free bond to protect the initial principal investment of an investor.

(iii). at t_0, $V_0 > P_0$.

(iv). If $Y_t > V_t$, Y_t is invested in a risky asset and the remaining $Y_t - V_t$ is borrowed. However, if $Y_t < V_t$, the investment in a risky asset does not need additional borrowing.

A CPPI strategy allows an investor to participate in the future returns of an investment by shifting financial resources between risky and risk-free asset classes according to a specific mechanical decision rule (Schied, 2014). When markets are rising an investor assumes high risk by shifting the portfolio towards a high-yielding risky asset. If the markets continue to rise, the dynamics of a CPPI strategy allow the investor to borrow some funds. The borrowed money enables a market participant to invest more in a risky asset than the total value of the portfolio. This indicates that the mechanics of CPPI techniques incorporate gearing. On the other hand, when markets are falling the market participant reduces an exposure in a risky asset by shifting more financial resources into a risk-free asset. In a worst case scenario in bearish markets, all money is placed in a risk-free asset to guarantee the initial principal investment. As a matter of fact, CPPI-based assets are called negative gamma products in the sense that, an investor buys a risky asset at a high price when markets are bullish and sell it at a low price when markets are bearish. This exercise goes against a common practice of buying an asset when its price is low and selling it when its price is high.

The remainder of this paper is organised as follows. The next Section is intended to introduce Preliminaries and Problem Formulation. Section 3 examines CPPI in Fuzzy Financial Markets. Finally, Conclusions are made in Section 4.

2. Preliminaries and Problem Formulation

From a mathematical perspective, the study considers a filtered credibility space $(\Theta, \mathcal{P}, \{\mathcal{F}_t\}_{t\geq 0}, \mathrm{Cr})$ equipped with a filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, generated by a one dimensional Liu process, $\{C_t\}_{t\geq 0}$, specified in the model.

Definition 2.1 (Liu, 2008) Suppose Θ is a non-empty set and $\mathcal{P}$ is a power set of Θ. Each element A in $\mathcal{P}$ is an event. A credibility measure is defined as a set function Cr: $\mathcal{P} \to [0,1]$ which satisfies the following four axioms:

- **Axiom 1:** (Normality) $\mathrm{Cr}\{\Theta\} = 1$;
- **Axiom 2:** (Monotonicity) $\mathrm{Cr}\{A\} \leq \mathrm{Cr}\{B\}$ if $A \subset B$;
- **Axiom 3:** (Self-Duality) $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1$ for any $A \in \mathcal{P}$;
- **Axiom 4:** (Maximality) $\mathrm{Cr}\{\cup_i A_i\} = \sup_i \mathrm{Cr}\{A_i\}$ for any events $\{A_i\}$ with $\sup_i \mathrm{Cr}\{A_i\} < 0.5$.

Definition 2.2 (Liu, 2008) A fuzzy variable is defined as a measurable function from a credibility space ($\Theta, \mathcal{P}$, Cr) to the set of real numbers.

Definition 2.3 (Liu, 2008) Suppose T is an index set and the triplet ($\Theta, \mathcal{P}$, Cr) is a credibility space. A fuzzy process is defined as a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.

Definition 2.4 (Liu, 2008) A fuzzy process C_t is a Liu process if

- $C_0 = 0$,
- C_t has stationary and independent increments,
- every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$, whose membership function is

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi |x - et|}{\sqrt{6}\sigma t}\right)\right)^{-1}, x \in \Re; \tag{1}$$

where e and σ are the drift and diffusion coefficients, respectively.

Definition 2.5 (Zhao and Liu, 2003) Suppose $\xi_1, \xi_2, \ldots$ are independent and identically distributed positive fuzzy inter-arrival times. By definition, $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \ldots + \xi_n$ for $n \geq 1$. Then the fuzzy process

$$N_t = \max_{n \geq 0}\{n \mid S_n \leq t\} \tag{2}$$

is called a fuzzy renewal process.

Definition 2.6 (Haugh, 2010) A filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, models the flow of information over a specific period of time. Given a credibility space (Θ, P, Cr), a filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, is an increasing family of σ-algebras on Θ, such that,

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \tag{3}$$

for $s \leq t$.

Definition 2.7 (Dai, 2007) Suppose V_t is a fuzzy process and P_t is a given level. Then the fuzzy variable

$$\tau_{P_t} = \inf\{t > 0 \mid V_t = P_t\} \tag{4}$$

is the first passage time that V_t touches the level P_t.

Financial markets are characterised by a multitude of indeterminate phenomena, for example, stock price. Indeterminacy is conceptualised as a condition of being unpredictable in advance as far as events' outcomes are concerned. It has already been demonstrated through empirical and theoretical research that risk is inherent in every investment undertaken because of indeterminacy in financial markets. Indeterminacy promotes the riskiness of financial markets.

Randomness, fuzziness and uncertainty are the three basic forms of indeterminacy in financial markets. In order to model these forms of indeterminacy, various mathematical systems have been developed. Probability theory (Kolmogorov, 1933) models randomness, fuzzy set theory (Zadeh, 1965) deals with fuzziness and uncertainty theory (Liu, 2007) models uncertainty. Several authors have analysed the dynamics of CPPI strategies using probability theory and uncertainty theory. The main aim of this research paper is to study the behaviour of CPPI strategies in fuzzy financial markets. This peace of research work examines the mechanics of CPPI techniques using credibility theory. Investors are interested in the effects of the participation rate on the CPPI-insured portfolio value and risk. In practice,

m is affected by factors such as limits on short sales, investor risk tolerance and limits on borrowing. The study explores the relationship between the participation rate and the CPPI-insured portfolio value. This research paper also seeks to scrutinize the relationship between m and the risk of the insured portfolio.

Empirical evidence indicates that it is a misconception to say that CPPI approaches always work. In practice, CPPI techniques are exposed to gap risk which emanates from sudden significant downward asset price jumps. Gap risk is the possibility that the CPPI-insured portfolio value may fall below a pre-determined floor. A CPPI-insured portfolio exhibits a loss when $V_t \leq P_t$. This research paper provides the basis for the determination of the participation rate based on investor's risk tolerance. The research paper also develops a strong foundation for the analytical computation of gap risk for CPPI strategies when asset price processes evolve as fuzzy differential equations with jumps. In terms of novelty, this study is the first peace of work to apply fuzzy calculus to CPPI.

3. CPPI in Fuzzy Financial Markets

The main goal of a CPPI strategy is to guarantee the initial principal investment at the end of the investment horizon. In order to scrutinize the dynamics of CPPI techniques in fuzzy financial markets, the study considers a market that comprises of two basic assets: stock (underlying risky asset) and default-free bond (riskless security).

CPPI in the Absence of Asset Price Jumps

At this juncture, the study assumes that asset prices follow a basic Liu stock model with constant interest rate r and constant volatility σ. The price process of the underlying risky asset, S_t, at time t evolves according to a geometric Liu process driven fuzzy differential equation given below

$$\frac{dS_t}{S_t} = \mu dt + \sigma dC_t, \tag{5}$$

where μ is the stock drift and σ is the stock diffusion, and C_t is a standard Liu process. The explicit solution of equation (5) is

$$S_t = S_0 \exp(\mu t + \sigma C_t). \tag{6}$$

On the other hand, the price process of a default-free bond, B_t, at time t is given by the following differential equation

$$\frac{dB_t}{B_t} = rdt, \tag{7}$$

where r is the riskless interest rate. The explicit solution of equation (7) is

$$B_t = B_0 \exp(rt). \tag{8}$$

Subsequently, μ is assumed to be greater than r.

Considering the dynamics of a CPPI strategy, the cushion satisfies the following fuzzy differential equation

$$\frac{dU_t}{U_t} = (m(\mu - r) + r)dt + m\sigma dC_t, \tag{9}$$

whose explicit solution is

$$U_t = U_0 \exp((m(\mu - r) + r)t + \sigma m C_t). \tag{10}$$

The mechanics of the cushion indicates that in a basic Liu stock model with continuous trading, the initial principal amount is always protected. In other words, in a basic Liu stock model with continuous trading, CPPI strategies are not exposed to gap risk. The basic understanding here is that when financial markets are falling the cushion eventually assumes a minimum value of zero. When markets fall, the cushion and the amount invested in the underlying risky asset also fall, and the converse is true. The dynamic shifting of financial resources between the risky and risk-free asset classes provides an initial principal amount guarantee at the end of the investment horizon.

The value of the CPPI-insured portfolio is described by the following formula

$$V_t = P_t + (V_0 - P_t e^{-rt})e^{(m(\mu - r) + r)t + \sigma m C_t}. \tag{11}$$

Equation (11) also supports the notion that in a Liu stock model with continuous trading, CPPI strategies always work. That is, in a Liu stock model environment, CPPI techniques are not exposed to gap risk whatever the participation rate.

The expected value of the CPPI-insured portfolio is independent of σ and is described by

$$E[V_t] = P_t + (V_0 - P_t e^{-rt}) e^{(m(\mu - r) + r)t}. \tag{12}$$

Formula (12) shows that, if $\mu > r$, the expected return of the insured portfolio can be infinitely increased, without assuming any additional risk, by taking higher and higher participation rates.

Summarily, in a basic Liu stock model environment, CPPI strategies are not exposed to gap risk, whatever the participation rate. This is attributed to the fact that a geometric Liu process is a continuous path process. In continuous time diffusion models stock prices are not exposed to jumps.. Therefore, the pay-off of a market participant at time T is denoted by

$$max\{V_T, B_T\} \equiv B_T + max\{U_T, 0\}. \tag{13}$$

In practice, however, stock prices exhibit jumps. The value of the CPPI-insured portfolio may fall below a specified lower bound in response to sudden significant downward asset price jumps, that is, CPPI strategies are exposed to gap risk. During this downward spiral in asset prices, the portfolio manager may not have enough time to readjust the portfolio before it falls below the floor. Illiquid underlying risky asset market magnifies jumps in asset market prices. The issuance of assets such as "Crash notes" in some financial markets provides enough evidence of the existence of asset price jumps. Crash notes are issued to hedge against downward jumps.

Proof of Gap Risk in CPPI Strategies

At the moment, the study provides a proof of the fact that CPPI strategies are exposed to gap risk. Until V_t reaches P_t at time τ the portfolio value evolves according to the following differential equation

$$dV_t = (U_t + P_t - mU_t)\frac{dB_t}{B_t} + mU_t \frac{dS_t}{S_t}, \tag{14}$$

which can be written as

$$\frac{dU_t}{U_t} = \frac{dB_t}{B_t} - m\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}, \tag{15}$$

$$\frac{dU_t}{U_t} = (1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}. \tag{16}$$

Suppose a fuzzy process dZ_t describes the relative change in the cushion, and is given by

$$dZ_t = (1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}. \tag{17}$$

Subsequently, equation (16) reduces to

$$\frac{dU_t}{U_t} = dZ_t. \tag{18}$$

This research work borrows the notion of a discounted cushion, $U_t^* = \frac{U_t}{B_t}$, from Cont and Tankov (2007). Applying the concept of a discounted cushion, the pay-off of an investor at t_0 is given by

$$1 + max\{U_T^*, 0\} \equiv 1 + max\{\frac{U_T}{B_T}, 0\}. \tag{19}$$

By introducing the discounted cushion $U_t^* = \frac{U_t}{B_t}$, equation (7) can be written as

$$U_t^* = U_0^* e^{Z_t}. \tag{20}$$

When the cushion falls, the amount invested in a risky asset also falls. All money is placed in a default-free bond when V_t touches P_t at time τ. As a result, the discounted cushion remain unchanged after time τ. The value of the discounted insured portfolio value is therefore described by

$$\frac{V_t}{B_t} = 1 + (\frac{V_0}{B_0} - 1)e^{Z_{t\wedge\tau}}. \tag{21}$$

Formula (21) shows that the exponential term can assume negative numbers when asset prices exhibit sudden significant downward shocks. This indicates that the initial principal amount is no longer guaranteed at the end of the investment period.

In a nutshell, CPPI strategies exhibit a loss when $V_t \le P_t$ or alternatively, $U_t \le 0$ or equivalently, $U_t^* \le 0$, for $\in [0,T]$. The following three events, $\{U_t^* \le 0\}$, $\{U_t \le 0\}$ and $\{\tau \le T\}$ indicate that gap risk for CPPI techniques is confronted during the investment horizon [0, T].

CPPI in the Presence of Jumps

Liu (2008) propounds that, in practice, stock prices are not continuous because of unexpected events and news such as wars and economic crises. In this section, the study seeks to analyse the dynamics of CPPI strategies in the presence of asset price jumps by adopting jump-diffusion models. Liu (2008) suggests a stock model with jumps. The stock price is assumed to follow a geometric Liu process with a jump. A fuzzy renewal process is said to be sufficient in order to model jumps.

The price process of a risky asset, S_t, at time t is described by the following fuzzy differential equation with jumps

$$\frac{dS_t}{S_t} = \mu dt + \sigma dC_t + \lambda dN_t, \tag{22}$$

where μ is the stock drift, σ is the stock diffusion, λ is the stock renewal coefficient, C_t is a standard Liu process and N_t is a renewal process.

Explicitly, the value of S_t at time t is given by

$$S_t = e^{\mu t + \sigma C_t + \lambda N_t}. \tag{23}$$

The price process of a default-free bond, B_t, at time t is described by the following differential equation

$$dB_t = B_t r dt, \tag{24}$$

where r is the risk-free interest rate. Conceptually, μ is assumed to be greater than r.

From the mechanics of a CPPI technique, the cushion satisfies the following fuzzy differential equation which is driven by a geometric Liu process and a renewal process

$$\frac{dU_t}{U_t} = (m(\mu - r) + r)dt + m\sigma dC_t + m\lambda dN_t, \tag{25}$$

whose explicit solution is

$$U_t = e^{(m(\mu - r) + r)t + m\sigma C_t + m\lambda N_t}. \tag{26}$$

Equation (25) shows that dU_t can become negative. Therefore, the initial principal amount, F, is not guaranteed at the end of the investment horizon, T.

In this case, the value of the CPPI-insured portfolio is described by

$$V_t = P_t + e^{(m(\mu - r) + r)t + m\sigma C_t + m\lambda N_t}. \tag{27}$$

Subsequently, the expected CPPI-insured portfolio value is given by

$$E[V_t] = P_t + e^{(m(\mu - r) + r)t + m\lambda N_t}. \tag{28}$$

A participation rate which is greater than one ($m > 1$) amplifies jumps in CPPI-insured portfolio value. When markets are rising, the higher the participation rate, the higher the rate at which CPPI-insured portfolio value increases. On the other hand, when markets are falling, the higher the participation rate, the higher the rate at which CPPI-insured portfolio value falls towards the floor. The volatility of the CPPI-insured portfolio value is proportional to m. Clearly, the risk of a loss in a CPPI strategy increases with m. In practice, it is not possible to continuously increase the CPPI-insured portfolio return by taking higher and higher participation rates without assuming any additional underlying risk. The direct relationship between m and V_t allows m to be gazetted based on the risk tolerance of the market participant.

The analysis of the dynamics of CPPI strategies develops a strong foundation for the analytical computation of gap risk for CPPI strategies when asset price processes evolve as fuzzy differential equations with jumps. Cont and Tankov (2007) propounds that m should be gazetted by relating it to some specified gap risk measures. Therefore, gap risk for CPPI strategies has to be quantified.

A basic Liu stock model is a continuous-path model which is popular because of its simplicity and analytical tractability. The modelling of risky asset prices by continuous-path models suffers from several defects which include the following

(i). The general assumption of path continuity is not practical considering the fact that asset prices do jump in response to factors such as time constraints, liquidity challenges, wars and market crashes. Generally, asset prices jump in response to unexpected events and news.

(ii). The modelling of underlying risky asset prices by a geometric Liu process is based on the use of a Gaussian distribution which underestimates the possibility of extreme events. Therefore, a geometric Liu process gives an approximation of the real behaviour of asset prices.

The available evidence indicates that a basic Liu stock model is not a good model of stock prices. A realistic asset price model needs to account for jumps in asset prices because

(i). Several risks cannot be modelled by continuous-path models.

(ii). Risk-neutral returns are regarded as non-Gaussian and leptokurtic. The law of returns for shorter investment horizons in continuous-path models is closer to the Gaussian law. However, in the real world and in models with jumps, asset returns are less Gaussian for shorter investment periods.

(iii). In reality, financial markets are incomplete and incomplete markets are generally associated with jump processes.

Therefore, a reasonable remedy is to use fuzzy processes which permit asset price jumps. Jump-diffusion models lead to more realistic modelling of asset prices. Fuzzy differential equations with jumps allow asset prices to jump whilst maintaining the independence and stationarity of returns. The mathematical tractability of jump-diffusion models makes it feasible to execute financial calculations and present their complicated results easily.

4. Conclusions

Several authors have analysed the dynamics of CPPI strategies using probability theory and uncertainty theory. This study is the first peace of research work to adopt credibility theory in examining the mechanics of CPPI strategies. In a basic Liu stock model environment, CPPI strategies always work, whatever the participation rate. If $\mu > r$, the expected return of the insured portfolio can be infinitely increased, without assuming any additional risk, by taking higher and higher participation rates. However, in practice, CPPI techniques are exposed to gap risk which emanates from sudden significant downward asset price shocks. A participation rate which is greater than one ($m > 1$) amplifies jumps in CPPI-insured portfolio value. Clearly, the risk of loss in a CPPI strategy increases with m. The direct relationship between m and V_t allows m to be gazetted based on the risk tolerance of the market participant. Gap risk for CPPI techniques is not insignificant. A multiplier should be gazetted by relating it to some specified gap risk measures. Therefore, gap risk for CPPI techniques has to be quantified. The analysis of the dynamics of CPPI strategies develops a strong foundation for the analytical computation of gap risk for CPPI strategies when asset price processes evolve as fuzzy differential equations with jumps. Adopting the jump-diffusion models whilst examining the mechanics of CPPI approaches maintains the analytical tractability of the continuous-time framework. In the next research paper the author seeks to develop analytical gap risk measures for CPPI strategies in fuzzy financial markets.

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