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Optimal Control Problems of Systems Governed by Elliptic Operators of Infinite Order with Pointwise Control Constraints

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Abstract

In this paper, we obtain the necessary and sufficient optimality conditions for elliptic control problem with pointwise control constraints generated by elliptic operators infinite order with finite dimensional and discussion of pointwise optimality conditions.

Keywords: Optimal Control of PDE, Infinite order operator, Pointwise control, Constrants.

1. Introduction

In Dubunski [1, 2, 3] studied the Cauchy Dirichlet problem

$$
\begin{aligned}
&L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma} u) = h(x), \qquad x \in G \\
&D^{|\omega|} u \,|_{\partial G} = 0, \qquad |\,\omega\,| = 0, 1, 2, \cdots
\end{aligned}
\tag{1}
$$

infinite order Sobolev spaces

$$
w^{\infty}\{a_{\alpha}, p_{\alpha}\} = \{u(x) \in C_0^{\infty}(G) : p(u) \equiv \sum_{|\alpha|=0}^{\infty} \|D^{\alpha} u\|_{p_{\alpha}}^{p_{\alpha}} < \infty\}
$$

where $a_{\alpha} \geq 0$ and $p_{\alpha} \geq 1$ are numerical sequences and established of $w^{\infty}\{a_{\alpha}, p_{\alpha}\}$ and boundary value problem (1) is investigated where $G \subset R^N$.

Gali et al. [10] presented a set of inequalities defining on optimal control of a system governed by self-adjoint elliptic operators with an infinite number of variables.

Subsequently Lions suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimensions.

Gali has solved this problem, the result has been published in [8]. I. M. Gali and S.-A. El-Saify and S. A. El-Zahaby [10,11,12] presented some control problems generated by both elliptic and hyperbolic linear operator of infinite order with finite number of variables.

El-Zahaby et al [4,5,6,7] obtained the optimal control of problems governed by variational inequalities of infinite order with bounded domain.

In this paper, we study an optimal control problem governed by a linear elliptic equation of infinite order operator with finite dimension.

Bounded constraints on the control are included in the formulation of the problem. The aim is to derive the necessary and sufficient first order conditions for optimality.

The paper is structured as follows:

- In section Two, introduce for functional spaces of infinite order with finite dimension.
- In section Three, we define the elliptic control problem generated by elliptic operator of infinite order with constraints on the control.
- In section Four, we derive the first order necessary conditions.
- In section Five, Discussion of point wise optimality conditions.

2. Some Functional Spaces [1-3]

The object of this section is to give the definition of some function spaces of infinite order and the chains of the constructed spaces which will be used later.

We define the Sobolev space $W^{\infty}\{a_\alpha, 2\}$ which shall denoted by $W^{\infty}\{a_\alpha, 2\}$ of infinite order of periodic functions $\varphi(x)$ defined on all boundary Γ of R^n, $n \geq 1$, as follows

$$W^{\infty}\{a_\alpha, 2\} = \{\varphi \in C^{\infty}(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha \varphi\|_2^2 < \infty\}$$

where $a_\alpha \geq 0$ is a numerical sequence and $\|\cdot\|_2$ is the canonical norm with space $L^2(R^n)$ all functions are assumed to be the real valued on

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots (\partial x_n^{\alpha_n})},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is a multi-index for differentiation $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

The space $W^{-\infty}\{a_\alpha, 2\}$ is defined as the formal conjugate space $W^{\infty}\{a_\alpha, 2\}$, namely:

$$W^{-\infty}\{a_\alpha, 2\} = \{\psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^{\alpha} \psi_\alpha(x)\}$$

where

$$\psi_\alpha \in L^2(R^2) \qquad \text{and} \qquad \sum_{|\alpha|=0}^{\infty} a_\alpha \| \psi_\alpha(x) \|_2^2 < \infty$$

the duality paining of the space $W^{\infty}\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ is postulated by the formula

$$(\varphi, \psi) = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{R^n} \psi_\alpha(x) D^{\alpha} \varphi(x) \, dx$$

where

$$\varphi \in W^{\infty}\{a_\alpha, 2\}, \qquad \psi \in W^{-\infty}\{a_\alpha, 2\}$$

From above, $W^{\infty}\{a_\alpha, 2\}$ is everywhere dense in $L^2(R^n)$ with topological inclusions and $W^{-\infty}\{a_\alpha, 2\}$ dense the topological dual space with respect to $L^2(R^n)$, so we have the following chain

$$W^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}$$

Analogous to the above chain we have

$$W_0^{\infty}\{a_\alpha, 2\} \subseteq L^2(R^n) \subseteq W_0^{-\infty}\{a_\alpha, 2\}$$

where $W_0^{\infty}\{a_\alpha, 2\}$ is the set of all function of $W^{\infty}\{a_\alpha, 2\}$ which vanish on the boundary Γ of R^n, i.e.,

$$W_0^{\infty}\{a_\alpha, 2\} = \{\phi(x) \in C_0^{\infty}(R^n) : \|\phi\|^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^{\alpha}\phi\|_2^2 < \infty,$$
$$D^{|w|}\phi|_{\Gamma} = 0, \ |w| = 0, 1, \cdots\}$$

Let us consider the elliptic operator of infinite order with finite dimension [8]

$$Ay = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} y \quad a_\alpha > 0. \tag{2}$$

This operator is bounded self-adjoint elliptic operator mapping $W_0^{\infty}\{a_\alpha, 2\}$ onto $W_0^{-\infty}\{a_\alpha, 2\}$.

We introduce a continuous bilinear form on $W_0^{\infty}\{a_\alpha, 2\}$

$$\begin{aligned}\pi(y,\phi) &= (Ay,\phi) \\ &= \sum_{|\alpha|=0}^{\infty} \left((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi(x)\right)_{L^2(R^n)},\ a_\alpha \ge 0 \\ &= \sum_{|\alpha|=1}^{\infty} \left((-1)^{|\alpha|} a_\alpha D^{2\alpha} y(x), \phi(x)\right)_{L^2(R^n)} + q(x)(y(x), \phi(x))_{L^2(R^n)}\end{aligned}$$

where $q(x)$ is a real valued function from $L_2(R^n)$ such that $q(x) \ge \nu$, $1 \ge \nu > 0$.

The ellipticity of A is sufficient from the coerciveness of $\pi(u,v)$ on $W^{\infty}\{a_\alpha, 2\}$, see [8]

$$\pi(u,u) \ge \nu \|u\|^2_{W^{\infty}\{a_\alpha,2\}}. \tag{3}$$

3. Optimal Control Problems for Infinite Order Type Equation

3.1 Problem statement

Assumption 1:

Let $\Omega \subset R^N$ be a bounded domain in R^N with Lipschtiz continuous boundary and suppose that $\lambda \ge 0$, $z_d \in L^2(\Omega)$, $\beta \in L^{\infty}(\Omega)$ for almost all $x \in \Gamma$, $u_a, u_b \in L^2(\Omega)$ with $u_a(x) \le u_b(x)$ f.a.a. $x \in E$. Here $\Omega = E$.

We consider the problem

$$\min J(y,u) := \frac{1}{2}\|y - z_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)} \tag{4}$$

Subject to

$$\begin{aligned} &Ay = \beta u \qquad \text{in } \Omega \\ &y^{|w|} = 0 \qquad \text{on } \Gamma, \quad |w| = 0,1,2,\cdots \text{on } \Gamma \end{aligned} \tag{5}$$

and

$$u_a(x) \le u(x) \le u_b(x) \qquad \text{f.a.a} \quad x \in \Omega \tag{6}$$

where A denotes an elliptic operator of infinite order having the form (2).

The function u denotes the control in the space $U = L^2(\Omega)$ and $y(u)$ is the solution (state of the function) associated to the control u.

Let us introduce the set of admissible control by

$$U_{ad} = \{u \in L^2(\Omega) : u_a(x) \le u(x) \le u_b(x) \text{ f.a.a } x \in \Omega\}$$

Note that $U_{ad} \subset U$ is non-empty, convex and bounded in $L^2(\Omega)$.

The following theorem follows from the Lax-Milgram Lemma and for a proof we refers to [13,14].

Theorem 3.1. *With Assumption 1 holding there exist a unique weak solution* $y \in W^{\infty}\{a_\alpha, 2\}$ *to* (5), *for every* $u \in L^2(\Omega)$ *i.e.*

$$\sum_{|\alpha|=1}^{\infty} \int_{R^n} (D^\alpha y)(x)(D^\alpha \varphi)(x)\,dx + \int_{R^n} q(x)y(x)\varphi(x)\,dx$$
$$= \int_{\Omega} \beta u \varphi\,dx \qquad \text{for all } \varphi \in W^{\infty}\{a_\alpha, 2\}$$

Furthermore

$$\|y\|_{W^{\infty}\{a_\alpha,2\}} \le C\|u\|_{L^2(\Omega)} \tag{7}$$

for a constant C *depending on* $\beta \in L^{\infty}(\Omega)$.

The unique solution y to (5) is called the state associated with u. We define the state space $Y = W^{\infty}\{a_\alpha, 2\}$ and we write $y = y(u)$ to emphasize the dependence u.

Definition 3.2. An element $\bar{u} \in U_{ad}$ is called optimal control and $\bar{y} = y(\bar{u})$ the associated optimal state provided

$$J(\bar{y}, \bar{u}) \le J(y(u), u) \qquad \text{forall} \quad u \in U_{ad} \tag{8}$$

The solution operator $G : L^2(\Omega) \to W^{\infty}\{a_\alpha, 2\}$, $u \to y(u)$ is well-defined by theorem 1. We call G the control-to-state mapping.

Notice that G is linear and continuous. The continuity follow from (7).

Remark 3.3. The space $W^{\infty}\{a_\alpha, 2\}$ and therefore $W_0^{\infty}\{a_\alpha, 2\} \subset W^{\infty}\{a_\alpha, 2\}$ is continuously embedded into $L_2(\Omega)$. In particular

$$\|y\|_{L^2(\Omega)} \le \|y\|_{W^\infty\{a_\alpha,2\}} \le C\|u\|_{L^2(\Omega)}$$

for $y = Gu$ and $u \in L^2(\Omega)$. Hence, we consider G as a mapping from $L^2(\Omega)$ to $L^2(\Omega)$. More precisely, we define the solution operator

$$S = E_Y G : L^2(\Omega) \to L^2(\Omega)$$

where

$$E_Y : W^\infty\{a_\alpha, 2\} \to L^2(\Omega)$$

denotes the canonical embedding operator. The advantage of the operator S is that its adjoint S^* is also defined on $L^2(\Omega)$ and we have operator S is that its adjoint S^* is also defined on $L^2(\Omega)$ and we have

$$(Su, \phi)_{L^2(\Omega)} = (u, S^*\phi)_{L^2(\Omega)} \quad \text{for all } \phi \in L^2(\Omega)$$

3.2 Existence of Optimal Control

For proving the existence of optimal controls, we transformed the control problems under investigation into reduced quadratic optimization problem in term of u, namely

$$\begin{aligned}\min_{u \in U_{ad}} f(u) &:= J(Su, u) \\ &= \frac{1}{2}\|Su - z_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}\end{aligned} \tag{9}$$

Theorem 3.4. *Let $U_{ad} \subset U = L^2(\Omega)$ be nonempty, bounded, closed and convex set and $z_d \in L^2(\Omega)$, $\lambda > 0$. The mapping S is assumed to be linear and continuous operator. Then there exists an optimal control $\overline{u}$ satisfy*

$$\min_{u \in U_{ad}} f(u) := \frac{1}{2}\|Su - z_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)}. \tag{10}$$

If $\lambda > 0$ hold or if S is injective, then $\overline{u}$ is uniquely determined.

Proof. Since $f(u) \ge 0$ holds, the infimum

$$j := \inf_{u \in U_{ad}} f(u)$$

exists. By assumption $U_{ad} \neq \phi$. Thus there is a minimizing sequence $\{u_n\}_{n \in N}$ satisfying

$$\lim_{n \to \infty} f(u_n) = j.$$

The set U_{ad} is bounded and closed (but in general not compact). From the convexity of U_{ad}, we infer that that U_{ad} is weakly sequentially compact. Thus, there exists subsequence $\{u_{nk}\}_{K \in N}$ of $\{u_n\}_{n \in N}$ converges weakly toan element $\overline{u} \in U_{ad}$ that is

$$u_{nk} \rightharpoonup \overline{u} \qquad \text{for} \quad k \to \infty$$

Since S is continuous, f is also continuous. From the convexity of f we find that f is weakly lower semicontinuous consequently,

$$f(\overline{u}) \le \liminf_{k \to \infty} f(u_{nk}) = j$$

Recall that j is the infimum of all function $f(u)$, $u \in U_{ad}$. From $\overline{u} \in U_{ad}$ we have $f(\overline{u}) \ge j$. Thus $f(\overline{u}) = j$ and $\overline{u}$ is an optimal control for (10).

Note that

$$f''(u) = S^* S + \lambda I : U \to U$$

is hessian of f where

$$S^* : L^2(\Omega) \to L^2(\Omega)$$

is the adjoint operator of S

$$S : L^2(\Omega) \to L^2(\Omega)$$

satisfying

$$(Su, h) = (u, S^* h) \quad \text{for all} \quad (u, h) \in U \times U$$

Notice that

$$(f''(u)v,v)_{L^2(\Omega)} = (S^*Sv+\lambda v,v)_{L^2(\Omega)}$$
$$=||Sv||^2+\lambda||v||^2$$

If $\lambda > 0$ then

$$(f''(u)v,v) > 0 \qquad \text{forall} \quad v \in U \setminus \{0\}.$$

On the other hand we have that S injective. Then $||Sv||^2_{L^2(\Omega)} > 0$ for all $v \in U \setminus \{0\}$. Thus, we have in both cases that $f''(u)$ is positive operator. This implies that f is strictly convex and there exists a unique optimal control.

4. Optimality Condition

We derive first order necessary optimality condition. Let U be a real Banach space, $\mathcal{U} \subset U = L^2(\Omega)$ be open, $C \subset \mathcal{U}$ be convex and $f : \mathcal{U} \to R$ a function which is Gateaux differentiable in $\mathcal{U}$. Suppose that $\bar{u} \in C$ is a solution to

$$\min_{u\in C} f(u) = \frac{1}{2}||Su - z_d||^2_{L^2(\Omega)} + \frac{\lambda}{2}||u||^2_{L^2(\Omega)}. \tag{11}$$

Then the following variational inequality holds

$$f'(\bar{u})(u-\bar{u}) \geq 0 \quad \text{for all} \quad u \in C \tag{12}$$

If $\bar{u} \in C$ solves (12) and f is convex, then $\bar{u}$ is a solution to (11) [13].

Theorem 4.1. *Let $U_{ad} \subset U = L^2(\Omega)$ be nonempty, convex and $z_d \in L^2(\Omega)$, $\lambda > 0$ be given. Furthermore assume that $S \in L(L^2(\Omega), L^2(\Omega))$. Then $\bar{u} \in U_{ad}$ solve* (11) *if the variational inequality*

$$(S^*(S\bar{u} - z_d) + \lambda\bar{u}, u - \bar{u})_{L^2(\Omega)} \geq 0 \quad \text{for all} \quad u \in U_{ad} \tag{13}$$

holds.

Proof. The gradient of f is given by

$$f'(\bar{u}) = S^*(S\bar{u} - z_d) + \lambda\bar{u}$$

then from (12) we have (13).

The variational inequality (13) can be expressed as

$$(S\overline{u} - z_d, Su - S\overline{u})_{L^2(\Omega)} + \lambda(\overline{u}, u - \overline{u})_{L^2(\Omega)} \geq 0 \quad \text{for all} \quad u \in U_{ad}$$

Definition 4.2. The weak solution $p \in W_0^\infty\{a_\alpha, 2\}$ of the adjoint or dual equation

$$\begin{aligned} Ap &= \overline{y} - z_d && \text{in} \quad \Omega \\ P^{|w|}|_\Gamma &= 0 && \text{on} \quad \Gamma \end{aligned} \tag{14}$$

with $\overline{y} = S\overline{u}$ is called the associated adjoint or dual state, let $z_d \in L^2(\Omega)$. Furthermore,

$$\overline{y} \in W_0^\infty\{a_\alpha, 2\} \to L^2(\Omega).$$

Thus $z_d - \overline{y}$ belong to $L^2(\Omega)$. By Lax-Miligrum there exist a unique state $p \in W_0^\infty\{a_\alpha, 2\}$ satisfying (14). Let y be the weak solution to problem (5).

Now, choosing p as is the weak formula of (5) we obtain

$$\sum_{|\alpha|=1}^{\infty} \int_\Omega a_\alpha (D^\alpha y)(x)(D^\alpha p)(x)\, dx + \int_\Omega q(x)y(x)p(x)\, dx = \int_\Omega \beta u p(x)\, dx.$$

On the other hand, For p we obtain the test function $y \in W_0^\infty\{a_\alpha, 2\}$ that

$$\sum_{|\alpha|=1}^{\infty} \int_\Omega a_\alpha (D^\alpha p)(x)(D^\alpha y)(x)\, dx + \int_\Omega q(x)p(x)y(x)\, dx = \int_\Omega (\overline{y} - z_d)y\, dx.$$

Since the left-hand sides are equal the assertion immediately follows and we find that

$$S^*(S\overline{u} - z_d) = S^*(\overline{y} - z_d) = \beta p.$$

Substitute in (13) we obtain

$$(\beta p + \lambda\overline{u}, u - \overline{u}) \geq 0 \qquad \forall\ u \in U_{ad}. \tag{15}$$

Theorem 4.3. *Suppose that* $\overline{u}$ *is an optimal control for the problem* (4)-(6) *and let* $\overline{y}$ *denote the associated state. Then the adjoint equation* (14) *has a unique weak solution* p *that satisfies the variational inequality*

$$\int_\Omega (\beta(x)p(x) + \lambda\overline{u}(x))(u(x) - \overline{u}(x))\, dx \geq 0 \quad \forall \quad u \in U_{ad} \tag{16}$$

Conversely any control $\overline{u} \in U_{ad}$ which, together with its associated state $\overline{y} = y(\overline{u}) = S\overline{u}$ and the solution p to (14) satisfies the variational inequality (16) is optimal solution to (4)-(6).

Summarizing, a control u is optimal for (4)-(6) if and only if u satisfies together with y and

p the following first-order necessary optimal system

$$\begin{aligned}
&Ay=\beta u && Ap=y-z_d\\
&y^{|w|}|_\Gamma=0,\quad |w|=0,1,2,\cdots && p^{|w|}|_\Gamma=0\\
&u\in U_{ad}\\
&(\beta p+\lambda\bar{u},v-\bar{u})_{L^2(\Omega)}\geq 0,\quad \text{for all}\quad v\in U_{ad}
\end{aligned}\tag{17}$$

5. Discussion of Pointwise Optimality Conditions

Next, we turn to a pointwise discussion of the optimality conditions from (17) we derive

$$\int_\Omega(\beta p+\lambda\bar{u})\bar{u}\ dx\leq\int_\Omega(\beta p+\lambda\bar{u})u\ dx\quad\forall\ u\in U_{ad}$$

hence

$$\int_\Omega(\beta p+\lambda\bar{u})\bar{u}\ dx=\min_{u\in U_{ad}}\int_\Omega(\beta p+\lambda\bar{u})u\ dx\quad\forall\ u\in U_{ad}\tag{18}$$

If $(\lambda\bar{u}+\beta P)$ is known (18) is a linear programming problem.

Lemma 5.1. *The variational inequality* (16) *holds if and only if for almost all* $x\in\Omega$*, we have*

$$\bar{u}(x)=\begin{cases}u_a(x), & \text{if}\quad \beta(x)p(x)+\lambda\bar{u}(x)>0;\\ \varepsilon\left[u_a(x),u_b(x)\right], & \text{if}\quad \beta(x)p(x)+\lambda\bar{u}(x)=0;\\ u_b(x), & \text{if}\quad \beta(x)p(x)+\lambda\bar{u}(x)<0.\end{cases}\tag{19}$$

The following pointwise variational inequality is equivalent to (19)

$$(\beta(x)p(x)+\lambda\bar{u}(x))(v-\bar{u}(x))\geq 0\quad\forall\ v\in[u_a(x),u_b(x)]\ \text{ for } a.e.\ x\in\Omega.\tag{20}$$

Proof.

(i) First, we show that (16) implies (19). We suppose that (19) does not hold and define the measurable sets

$$\begin{aligned}
A_+(\bar{u})&=\{x\in\Omega:\beta(x)p(x)+\lambda\bar{u}(x)>0\},\\
A_-(\bar{u})&=\{x\in\Omega:\beta(x)p(x)+\lambda\bar{u}(x)<0\}.
\end{aligned}$$

Analogously, u_a and u_b stand for arbitrary but fixed representant the claim hold also for any chosen representants.

By assumption (19) is not satisfied. Thus there exists a set $E_+\subset A_+(\bar{u})$ with positive measure and

$$\bar{u}(x) > u_a(x) \quad \text{forall} \quad x \in E_+$$

or a set $E_- \subset A_-(\bar{u})$ with positive measure and

$$\bar{u}(x) < u_a(x) \quad \text{forall} \quad x \in E_-$$

let the function $u \in U_{ad}$,

$$u(x) = \begin{cases} u_a(x), & \text{for} \quad x \in E_+; \\ \bar{u}(x), & \text{for} \quad x \in \Omega \setminus E_+. \end{cases}$$

Then

$$\int_\Omega (\beta(x)p(x) + \lambda\bar{u}(x))(u(x) - \bar{u}(x))\ dx =$$
$$\int_{E_+} (\beta(x)p(x) + \lambda\bar{u}(x))(u_a(x) - \bar{u}(x))\ dx < 0$$

Since

$$\lambda\bar{u} + \beta p > 0$$

and

$$u_a(x) < \bar{u}(x) \qquad \text{on} \qquad E_+ \subset A_+(\bar{u})$$

Since $u \in U_{ad}$ holds, we have a contradiction to (16). The other case can be handled in asimilar way by putting

$$u(x) = u_b(x) \qquad \text{on} \quad E_-$$

and $u(x) = \bar{u}(x)$ otherwise.

(ii) Next, we show that (19) implies (20): We have

$$\bar{u}(x) = u_a(x) \qquad \text{on} \quad A_+(\bar{u}) \quad \text{a.e.}$$

Thus

$$v - \bar{u}(x) \geq 0 \qquad \text{forall} \qquad v \in [u_a(x), u_b(x)] \quad \text{for} \quad x \in A_+(\bar{u}) \quad \text{a.e.}$$

Utilizing

$$(\beta p(x) + \lambda \bar{u}(x))(v - \bar{u}(x)) \geq 0 \quad \text{in} \quad A_+(\bar{u}) \quad \text{a.e.}$$

Analogously, we derive

$$(\beta p(x) + \lambda \bar{u}(x))(v - \bar{u}(x)) \geq 0 \quad \text{in} \quad A_-(\bar{u}) \quad \text{a.e.}$$

Clearly (20) holds on

$$\Omega \setminus (A_+(\bar{u}) \cup A_-(\bar{u})) \qquad \text{a.e.}$$

(iii) Finally, we show that (20)- implies (16): let $u \in U_{ad}$ be chosen arbitrary we have

$$\bar{u}(x) \in [u_a(x), u_b(x)] \quad \text{f.a.a} \quad x \in \Omega.$$

Using (20) with $v = u(x)$ we have

$$(\beta p(x) + \lambda \bar{u}(x))(u(x) - \bar{u}(x)) \geq 0 \quad \text{f.a.a} \quad x \in \Omega.$$

By integrating (16) follows immediately.

From (20) we deduce

$$(\beta p(x) + \lambda \bar{u}(x))\bar{u}(x) \leq (\beta p(x) + \lambda \bar{u}(x))v \quad \text{forall} \quad v \in [u_a(x), u_b(x)] \tag{21}$$

From the chain for the regularization parameter λ we can deduce further consequences.

Case 1: $\lambda = 0$. using (19) we find

$$\bar{u}(x) = \begin{cases} u_a(x), & \text{if} \quad \beta(x)p(x) > 0; \\ u_b(x), & \text{if} \quad \beta(x)p(x) < 0. \end{cases}$$

If $\beta(x)p(x) = 0$ holds, we do not get any information for $\bar{u}(x)$. In this case

$$\beta(x)p(x) \neq 0 \quad \text{f.a.a} \qquad x \in \Omega$$

we have

$$\bar{u}(x) = u_a(x) \quad \text{or} \quad \bar{u}(x) = u_b(x) \quad \text{f.a.a} \qquad x \in \Omega$$

In this case we have a so-called bang-bang control.

Case 2: $\lambda > 0$, we derive from (19) that

$$\bar{u}(x) = \frac{-1}{\lambda}(\beta(x)p(x))$$

holds if

$$\beta(x)p(x)+\lambda\bar{u}(x)=0.$$

This leads to the following theorem.

Theorem 5.2. *Let* $\lambda>0$. *Then* $\bar{u}$ *is a solution to* (4) *if and only if*

$$\bar{u}(x)=P_{[u_a(x),u_b(x)]}\left(\frac{-1}{\lambda}\beta(x)p(x)\right)$$

where $P_{[a,b]}$, $a<b$, *is the projection of* R *on* $[a,b]$ *given by*

$$P_{[a,b]}u:=\min(b,\max(a,u))$$

Case $\lambda>0$ and $U_{ad}=L^2(\Omega)$ (no control constraints). From (16) or (21) it follows directly

$$\bar{u}=-\frac{1}{\lambda}\beta p. \tag{22}$$

This we obtain the following optimality system

$$Ay=-\frac{1}{\lambda}\beta^2 p, \qquad Ap=y-z_d$$

$$y^{|w|}|_\Gamma=0,\quad |w|=0,1,2,\cdots \qquad p^{|w|}|_\Gamma=0\quad |w|=0,1,2,\cdots$$

Which is a coupled system of two elliptic equations. If p is computed, $\bar{u}$ is given by (22).

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