"Science Stays True Here"
Journal of Mathematics and Statistical Science, Volume 2015, 167-198 | Science Signpost Publishing

An Extended Lindley Poisson Distribution with Applications

Mavis Pararai

Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA, 15705, USA.

Gayan Warahena-Liyanage

Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA, 15705, USA.

Broderick O. Oluyede

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA, 30460, USA.

Abstract

The Extended Lindley Poisson (ELP) distribution which is an extension of the extended Lindley distribution [2] is introduced and its properties are explored. This new distribution represents a more flexible model for the lifetime data. Some statistical properties of the proposed distribution including the shapes of the density, hazard rate functions, moments, Bonferroni and Lorenz curves are explored. Entropy measures and the distribution of the order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters and a simulation study is conducted to investigate the performance of the maximum likelihood estimates. Finally, we present applications of the model with a real data set to illustrate the usefulness of the proposed distribution.

Keywords: Lindley distribution, Lindley Poisson distribution, Lifetime data, Maximum likelihood estimation.

1. Introduction

Lindley [10] developed Lindley distribution in the context of fiducial and Bayesian statistics. Properties, extensions and applications of the Lindley distribution have been studied in the context of reliability analysis by Ghitany et al. [8], Zakerzadeh and Dolati [25], and Warahena-Liyanage and Pararai

[24]. Several authors including Sankaran [20], Asgharzadeh et al. [1] and Nadarajah et al. [15] proposed and developed the mathematical properties of various generalized Lindley distributions. Pararai et al. [18] studied the properties of the beta exponentiated power Lindley distribution which is a generalization of the beta generalized Lindley distribution by Oluyede and Tiantin [16]. Using the transformation $X = Y^{1/\alpha}$, Ghitany et al. [9] developed and studied the properties of the power Lindley distribution.

The cumulative distribution function (cdf) and the corresponding probability density function (pdf) of the one-parameter Lindley distribution [10] are given by

$$F(x;\lambda) = 1 - \left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)e^{-\lambda x}, \tag{1.1}$$

and

$$f(x;\lambda) = \frac{\lambda^2}{\lambda+1}(1+x)e^{-\lambda x}, \tag{1.2}$$

for $x > 0$, $\lambda > 0$, respectively. The Lindley distribution has been compounded with other distributions such as geometric, Poisson and logarithmic to develop new families of continuous lifetime distributions. The transmuted Lindley-geometric distribution was studied by Merovci and Elbatai, [11]. Warahena-Liyanage and Pararai [24] studied the Lindley-logarithmic distribution and its properties in detail. The Lindley-Poisson which is a submodel of the exponentiated power Lindley Poisson distribution was studied by Pararai et al. [18]. The Lindley-geometric distribution was studied by Zakerzadeh and Mahmoudi [26]. Bakouch et al. [2] derived and studied the properties of the extended Lindley distribution by considering a particular exponentiation. The cdf and pdf of the three-parameter extended Lindley distribution are given by

$$F(x;\alpha,\beta,\lambda) = 1 - \left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha} e^{-(\lambda x)^{\beta}}, \tag{1.3}$$

and

$$f(x;\alpha,\beta,\lambda) = \frac{\lambda(1+\lambda+\lambda x)^{\alpha-1}}{(1+\lambda)^{\alpha}}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1} - \alpha\right]e^{-(\lambda x)^{\beta}}, \tag{1.4}$$

for $x > 0, \alpha \in \mathbb{R}^{-} \cup \{0,1\}, \lambda > 0,$ and $\beta \geq 0$. When $\alpha = \beta = 1,$ the extended Lindley distribution (EL) becomes the Lindley distribution.

Motivated by the advantages of the Lindley distribution with respect to having a hazard function that exhibits increasing, decreasing and bathtub shapes, as well as the versatility and flexibility of compounding Lindley with other distributions such as geometric, logarithmic and Poisson distributions in modeling lifetime data, we propose and study a new distribution called the extended Lindley-Poisson (ELP) distribution, which inherits these desirable properties that also cover the shapes of quite a large number of models.

This paper is organized as follows: In section 2; the ELP model, its sub-models and some statistical properties such as quantile function and hazard function are presented. In section 3; we present the moments. Section 4 contains the distribution of the order statistics and entropy measures. Mean deviations, Bonferonni and Lorenz curves are presented in section 5. Maximum likelihood estimates of the model parameters and asymptotic confidence intervals are given in section 6. A simulation study is also presented in section 6. Section 7 contains applications of the proposed model to real data, followed by concluding remarks in section 8.

2. The Model, Sub-Models and Some Properties

Suppose that the random variable X has the extended Lindley distribution where its cdf and pdf are given in equations (1.3) and (1.4). Given N, let $X_1,...,X_N$ be independent and identically distributed random variables from Lindley distribution. Let N be distributed according to the zero truncated Poisson distribution [5] with pdf

$$P(N=n)=\frac{\theta^n e^{-\theta}}{n!(1-e^{-\theta})},\quad n=1,2,...,\theta>0.$$

Let X = max $(Y_1,...,Y_N)$, then the cdf of $X\mid N=n$ is given by

$$G_{X|N=n}(x)=\left[1-\left(\frac{1+\lambda+\lambda x}{1+\lambda}\right)^{\alpha} e^{-(\lambda x)^{\beta}}\right]^{n},$$

for $x>0,\ \ \alpha\in\mathbb{R}^{-}\cup\{0,1\},\ \ \lambda>0,\ \ \beta>0,$ which is the exponentiated extended Lindley distribution. The extended Lindley-Poisson (ELP) distribution denoted by ELP $(\alpha,\beta,\lambda,\theta)$ is defined by the marginal cdf of X, that is,

$$G_{ELP}(x;\alpha,\beta,\lambda,\theta)=\frac{1-\exp\left\{\theta\left[1-\left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha}e^{-(\lambda x)^{\beta}}\right]\right\}}{1-e^{\theta}}, \tag{2.1}$$

for $x>0$, $\alpha\in\mathbb{R}^{-}\cup\{0,1\}$, $\lambda>0$, $\theta>0$ and $\beta\geq 0$. Plots of the cdf for the ELP distribution for several values of the parameters α, β, λ and θ are given in Figure 2.1.

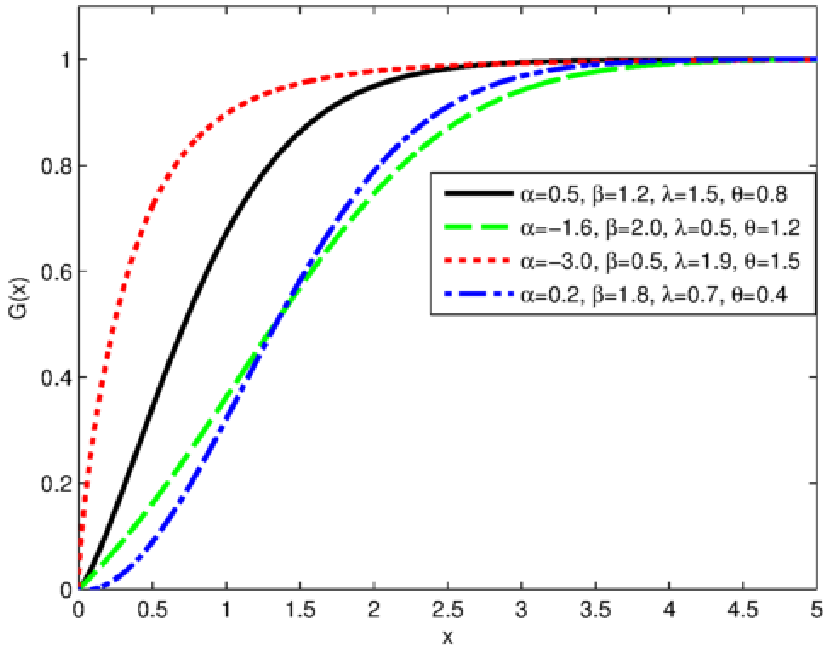

Figure 2.1. Plot of the CDF for different values of α, β, λ and θ

The pdf of the ELP distribution is given by

$$g_{ELP}(x,\alpha,\beta,\lambda,\theta)=\frac{\theta\lambda(1+\lambda+\lambda\alpha)^{\alpha-1}}{(1+\lambda)^{\alpha}(e^{\theta}-1)}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]e^{-(\lambda x)^{\beta}}$$
$$\times\exp\left\{\theta\left[1-\left(\frac{1+\lambda+\lambda x}{1+\lambda}\right)^{\alpha}e^{-(\lambda x)^{\beta}}\right]\right\}, \tag{2.2}$$

for $x>0$, $\alpha\in\mathbb{R}^{-}\cup\{0,1\}$, $\lambda>0$, $\theta>0$ and $\beta\geq 0$. Plots for the pdf of ELP distribution for several values of the parameters λ, θ, α and β are given Figure 2.2.

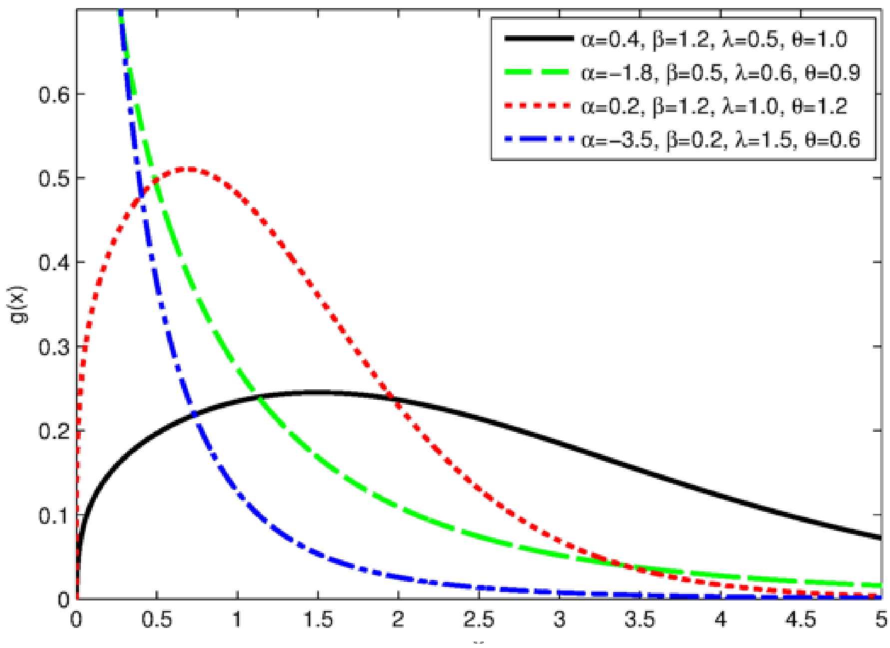

Figure 2.2. Plot of the PDF for different values of λ, θ, α and β

2.1 Sub-models

Some sub-models of the ELP distribution are presented in this section.

- When $\alpha = \beta = 1$, we get the Lindley-Poisson (LP) distribution, (Pararai et. al, [17]) distribution whose cdf is

$$G(x) = \frac{1 - \exp\left\{\theta\left[1 - \left(\frac{1+\lambda+\lambda x}{1+\lambda}\right)e^{-\lambda x}\right]\right\}}{1 - e^{\theta}}$$

- When $\alpha = 0$, we obtain the Weibull-Poisson (WP), (Lu and Shi [14]) distribution whose cdf is given by

$$G(x) = \frac{1 - \exp\left\{\theta\left[1 - e^{(-\lambda x)^{\beta}}\right]\right\}}{1 - e^{\theta}}$$

- When $\alpha = 0$, and $\beta = 1$, we obtain the exponential Poisson (EP), (Tahmasbi and Rezaei [23]), whose cdf is given by

$$G(x) = \frac{1 - \exp\left\{\theta\left[1 - e^{-\lambda x}\right]\right\}}{1 - e^{\theta}}$$

- When $\alpha = 0$, and $\beta = 2$, we obtain the Rayleigh-Poisson (RP) distribution whose cdf is given by

$$G(x) = \frac{1 - \exp\left\{\theta\left[1 - e^{(-\lambda x)^2}\right]\right\}}{1 - e^{\theta}}$$

- When $\alpha = \beta = 1$, and $\theta \to 0^+$, we obtain the Lindley (L) distribution [10].

2.2 Quantile Function

The quantile function is the solution of the equation

$$G(x_p) = \frac{1 - \exp\left\{\theta\left[1 - \left(\frac{1+\lambda+\lambda x_p}{\lambda+1}\right)^{\alpha} e^{-(\lambda x_p)^{\beta}}\right]\right\}}{1 - e^{\theta}} = p,$$

where $0 < p < 1$.

Thus, we have

$$x_p = \left[\frac{\alpha}{\lambda^{\beta}} \ln\left(\frac{1 + \lambda + \lambda x_p}{1 + \lambda(1-q)^{1/\alpha}}\right)\right]^{1/\beta}, \tag{2.3}$$

where $q = \frac{\ln(1 - p(1 - e^{\theta}))}{\theta}$ and p is uniformly distributed on the interval $(0,1)$.

2.3 Hazard and Reverse Hazard Functions

The survival function for the ELP distribution is given by,

$$\begin{aligned}\overline{G}(x;\lambda,\theta,\alpha,\beta) &= 1 - G(x;\lambda,\theta,\alpha,\beta)\\ &= \frac{\exp\left\{\theta\left[1 - \left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha} e^{-(\lambda x)^{\beta}}\right]\right\} - e^{\theta}}{1 - e^{\theta}}.\end{aligned} \tag{2.4}$$

The hazard and reverse hazard functions are given by

$$\lambda_G(x;\lambda,\theta,\alpha,\beta)=\frac{g(x;\lambda,\theta,\alpha,\beta)}{\bar{G}(x;\lambda,\theta,\alpha,\beta)}$$

$$=\frac{\theta\lambda(1+\lambda+\lambda x)^{\alpha-1}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]e^{-(\lambda x)^{\beta}}}{(1+\lambda)^{\alpha}\left[e^{\theta}-e^{\theta\left[1-\left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha}e^{-(\lambda x)^{\beta}}\right]}\right]}$$

$$\times\exp\left\{\theta\left[1-\left(\tfrac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha}e^{-(\lambda x)^{\beta}}\right]\right\},$$

and

$$\tau_G(x;\lambda,\theta,\alpha,\beta)=\frac{g(x;\lambda,\theta,\alpha,\beta)}{G(x;\lambda,\theta,\alpha,\beta)}$$

$$=\frac{\theta\lambda(1+\lambda+\lambda x)^{\alpha-1}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]e^{-(\lambda x)^{\beta}}}{(1+\lambda)^{\alpha}\left[e^{\theta\left[1-\left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha}e^{-(\lambda x)^{\beta}}\right]}-1\right]}$$

$$\times\exp\left\{\theta\left[1-\left(\tfrac{1+\lambda+\lambda x}{\lambda+1}\right)^{\alpha}e^{-(\lambda x)^{\beta}}\right]\right\},$$

respectively. Plots of the hazard function are given below:

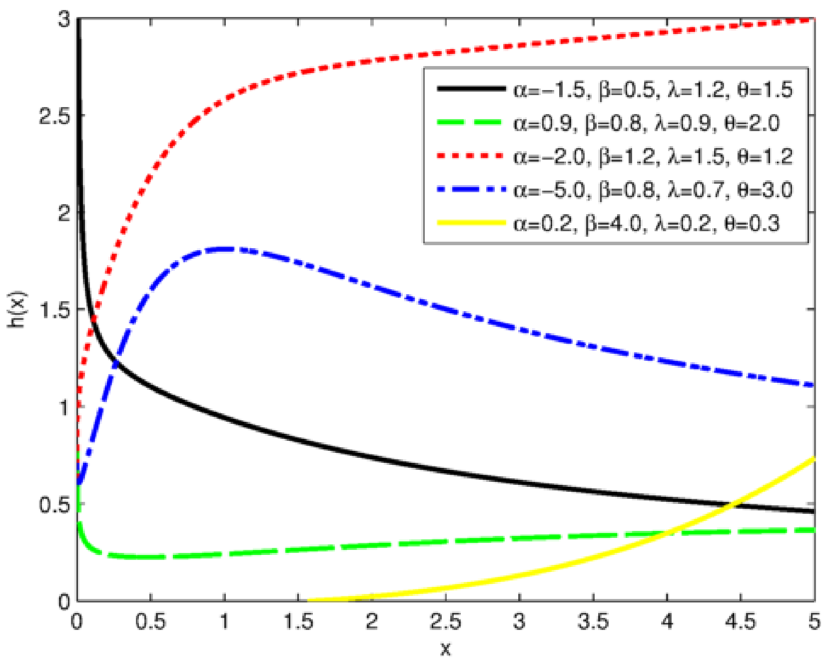

Figure 2.3. Plot of the hazard function for different values of λ,θ,α and β

The graphs of the hazard function for four combinations of the values of the model parameters show various shapes including monotonically increasing, monotonically decreasing, uni-modal, bathtub, and upside down bathtub shapes with four combinations of the values of the parameters. This attractive flexibility makes the ELP hazard rate function useful and suitable for non-monotone empirical hazard behaviors which are more likely to be encountered or observed in real life situations.

3. Moments, Moment Generating Function and Related Measures

In this section, moments and related measures including coefficients of variation, skewness and kurtosis are presented. A table of values for mean,standard deviation, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) is also presented.

3.1 Moments

The r^{th} moment of the ELP distribution is given by $E(X^r)=\int_0^\infty x^r g(x)dx$. Thus,

$$\begin{aligned}E(X^r) &= \int_0^\infty x^r g(x)dx \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j}{k}\frac{(-1)^k\theta^{j+1}\lambda}{(1+\lambda)^{\alpha k+\alpha}(e^{\theta}-1)j!} \\ &\times\left[\beta\sum_{p=0}^{\infty}\binom{\alpha k+\alpha}{q}(1+\lambda)^{\alpha k+\alpha-p}\lambda^{p+\beta-1}\int_0^\infty x^{r+p+\beta-1}e^{-(k+1)(\lambda x)^{\beta}}dx\right. \\ &\left.-\alpha\sum_{q=0}^{\infty}\binom{\alpha k+\alpha-1}{q}(1+\lambda)^{\alpha k+\alpha-q-1}\lambda^{q}\int_0^\infty x^{r+q}e^{-(k+1)(\lambda x)^{\beta}}dx.\right]\end{aligned} \tag{3.1}$$

Let $u=(k+1)(\lambda x)^{\beta}$, then $du=\beta(k+1)\lambda^{\beta}x^{\beta-1}dx$ and $x=\frac{u^{\frac{1}{\beta}}}{\lambda(k+1)^{\frac{1}{\beta}}}$. Consider

$$\begin{aligned}\beta\lambda^{p+\beta-1}\int_0^\infty x^{r+p+\beta-1}e^{-(k+1)(\lambda x)^{\beta}}dx &= \frac{\lambda^{p-1}}{k+1}\int_0^\infty x^{r+p}e^{-(k+1)(\lambda x)^{\beta}} \\ &\times\beta(k+1)\lambda^{\beta}x^{\beta-1}dx \\ &= \frac{1}{\lambda^{r+1}(k+1)^{\frac{r+p}{\beta}+1}}\int_0^\infty u^{\frac{r+p}{\beta}}e^{u}du. \\ &= \frac{\Gamma(\frac{r+p}{\beta}+1)}{\lambda^{r+1}(k+1)^{\frac{r+p}{\beta}+1}}.\end{aligned} \tag{3.2}$$

Similarly,

$$
\begin{aligned}
\int_0^\infty x^{q+r} e^{-(k+1)(\lambda x)^\beta} dx &= \frac{1}{\beta\lambda^\beta (k+1)} \int_0^\infty x^{r+q-\beta+1} e^{-(k+1)(\lambda x)^\beta} \\
&\quad \times \beta(k+1)\lambda^\beta x^{\beta-1} dx \\
&= \frac{1}{\beta\lambda^\beta (k+1)} \int_0^\infty \frac{u^{\frac{r+q+1}{\beta}-1}}{\lambda^{r+q-\beta+1}(k+1)^{\frac{q+r+1}{\beta}-1}} \\
&= \frac{\Gamma(\frac{r+q+1}{\beta})}{\beta\lambda^{q+r+1}(k+1)^{\frac{r+q+1}{\beta}}}.
\end{aligned} \tag{3.3}
$$

Thus the r^{th} moment is given by

$$
\begin{aligned}
E(X^r) &= \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{j}{k} \frac{(-1)^k \theta^{j+1}\lambda}{(1+\lambda)^{\alpha k+\alpha}(e^\theta - 1) j!} \\
&\times \left[\sum_{p=0}^\infty \binom{\alpha k+\alpha}{q} \frac{(1+\lambda)^{\alpha k+\alpha-p}\Gamma(\frac{r+p}{\beta}+1)}{\lambda^{r+1}(k+1)^{\frac{r+p}{\beta}+1}} \right. \\
&\left. -\alpha \sum_{q=0}^\infty \binom{\alpha k+\alpha-1}{q} \frac{(1+\lambda)^{\alpha k+\alpha-q-1}\Gamma(\frac{r+p+1}{\beta})}{\beta\lambda^{q+r+1}(k+1)^{\frac{r+p+1}{\beta}}} \right].
\end{aligned} \tag{3.4}
$$

The mean, variance, coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$
\begin{aligned}
\mu = \mu_1' &= \sum_{j=0}^\infty \sum_{k=0}^\infty \binom{j}{k} \frac{(-1)^k \theta^{j+1}\lambda}{(1+\lambda)^{\alpha k+\alpha}(e^\theta - 1) j!} \\
&\times \left[\sum_{p=0}^\infty \binom{\alpha k+\alpha}{q} \frac{(1+\lambda)^{\alpha k+\alpha-p}\Gamma(\frac{r+p}{\beta}+1)}{\lambda^{2}(k+1)^{\frac{r+p}{\beta}+1}} \right. \\
&\left. -\alpha \sum_{q=0}^\infty \binom{\alpha k+\alpha-1}{q} \frac{(1+\lambda)^{\alpha k+\alpha-q-1}\Gamma(\frac{q+2}{\beta})}{\beta\lambda^{q+2}(k+1)^{\frac{q+2}{\beta}}} \right].
\end{aligned} \tag{3.5}
$$

$$
\sigma^2 = \mu_2' - \mu^2, \tag{3.6}
$$

$$
CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1}, \tag{3.7}
$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}, \tag{3.8}$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^{2}} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^{2}}, \tag{3.9}$$

respectively.

Table 3.1 lists the first six moments of the ELP distribution for selected values of the parameters by fixing $\theta = 1.5$ and $\lambda = 0.5$. These values can be determined numerically using R and MATLAB.

3.2 Moment Generating Function

The moment generating function of the ELP dustribution is given by

$$\begin{aligned} E(e^{tX}) &= \sum_{b=0}^{\infty} \frac{t^b}{b!} E(X^b) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{b=0}^{\infty} \binom{j}{k} \frac{(-1)^k \theta^{j+1} t^b \lambda}{(1+\lambda)^{\alpha k+\alpha}(e^\theta - 1) j! b!} \\ &\times \left[\sum_{p=0}^{\infty} \binom{\alpha k+\alpha}{p} \frac{(1+\lambda)^{\alpha k+\alpha-p}\Gamma(\frac{b+p}{\beta}+1)}{\lambda^{b+1}(k+1)^{\frac{b+p}{\beta}+1}} \right. \\ &\left. -\alpha \sum_{q=0}^{\infty} \binom{\alpha k+\alpha-1}{q} \frac{(1+\lambda)^{\alpha k+\alpha-q-1}\Gamma(\frac{b+p+1}{\beta})}{\beta\lambda^{q+b+1}(k+1)^{\frac{b+p+1}{\beta}}} \right]. \end{aligned} \tag{3.10}$$

3.3 Distribution of Order Statistics

Order Statistics play a vital role in probability and statistics. In this section, we present the distribution of the order statistics for the EPL distribution. The pdf of the i^{th} order statistic is given by:

$$
\begin{aligned}
g_{i:n}(x) &= \frac{n!g(x)}{(i-1)!(n-i)!}[G(x)]^{i-1}[1-G(x)]^{n-i} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{r=0}^{\infty}(-1)^r\binom{n-i}{r}[G(x)]^{i+r-1} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{r=0}^{\infty}(-1)^r\binom{n-i}{r}\frac{\left[1-\exp\left\{\theta\left[1-\left(1+\frac{\lambda y}{\lambda+1}\right)^{\alpha}e^{-(\lambda y)^{\beta}}\right]\right\}\right]^{i+r-1}}{\left[1-e^{\theta}\right]^{i+r-1}} \\
&= \frac{n!g(x)}{(i-1)!(n-i)!}\sum_{r,s=0}^{\infty}(-1)^{r+s}\binom{n-i}{r}\binom{i+r-1}{s}\frac{\exp\left\{\theta s\left[1-\left(1+\frac{\lambda y}{\lambda+1}\right)^{\alpha}e^{-(\lambda y)^{\beta}}\right]\right\}}{\left[1-e^{\theta}\right]^{i+r-1}} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{r,s=0}^{\infty}(-1)^{s+1-i}\binom{n-i}{r}\binom{i+r-1}{s}\frac{\theta\lambda(1+\lambda+\lambda x)^{\alpha-1}}{(1+\lambda)^{\alpha}\left[e^{\theta}-1\right]^{i+r}} \\
&\times\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]e^{-(\lambda y)^{\beta}}\exp\left\{\theta(s+1)\left[1-\left(1+\frac{\lambda y}{\lambda+1}\right)^{\alpha}e^{-(\lambda y)^{\beta}}\right]\right\} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{r,s,t=0}^{\infty}(-1)^{s+1-i}\binom{n-i}{r}\binom{i+r-1}{s}\frac{\lambda(1+\lambda+\lambda x)^{\alpha-1}}{(1+\lambda)^{\alpha}\left[e^{\theta}-1\right]^{i+r}t!} \\
&\times\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]e^{-(\lambda y)^{\beta}}\theta^{t+1}(s+1)^{t}\left[1-\left(1+\frac{\lambda y}{\lambda+1}\right)^{\alpha}e^{-(\lambda y)^{\beta}}\right]^{t} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{r,s,t,v=0}^{\infty}(-1)^{s+v+1-i}\binom{n-i}{r}\binom{i+r-1}{s}\binom{t}{v} \\
&\times\frac{\lambda\theta^{t+1}(s+1)^{t}(1+\lambda+\lambda x)^{\alpha v+\alpha-1}}{(1+\lambda)^{\alpha+v}\left[e^{\theta}-1\right]^{i+r}t!}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]e^{-(v+1)(\lambda x)^{\beta}} \\
&= \frac{n!}{(i-1)!(n-i)!}\sum_{r,s,t,v=0}^{\infty}(-1)^{s+v+1-i}\binom{n-i}{r}\binom{i+r-1}{s}\binom{t}{v} \\
&\times\frac{\lambda\theta^{t+1}(s+1)^{t}(1+\lambda+\lambda x)^{\alpha v+\alpha-1}}{(1+\lambda)^{\alpha+v}\left[e^{\theta}-1\right]^{i+r}t!}\left[\beta\sum_{d=0}^{\infty}\binom{\alpha v+\alpha}{d}(1+\lambda)^{\alpha v+\alpha-d}(\lambda x)^{\beta+d-1}\right. \\
&\left.-\alpha\sum_{f=0}^{\infty}\binom{\alpha v+\alpha-1}{f}(1+\lambda)^{\alpha v+\alpha-f-1}(\lambda x)^{f}\right]e^{-(v+1)(\lambda x)^{\beta}}.
\end{aligned}
\tag{3.11}
$$

The m^{th} moment of the i^{th} order statistic is given by

$$E(X_{i:n}^m)=\frac{n!}{(i-1)!(n-i)!}\sum_{r,s,t,v=0}^{\infty}(-1)^{s+v+1-i}\binom{n-i}{r}\binom{i+r-1}{s}\binom{t}{v}$$
$$\times\frac{\lambda\theta^{t+1}(s+1)^t(1+\lambda+\lambda x)^{\alpha v+\alpha-1}}{(1+\lambda)^{\alpha+v}[e^{\theta}-1]^{i+r}t!}$$
$$\times\left[\sum_{d=0}^{\infty}\binom{\alpha v+\alpha}{d}(1+\lambda)^{\alpha v+\alpha-d}\frac{\Gamma\left(\frac{m+d+\beta}{\beta}\right)}{(v+1)^{\frac{m+d+\beta}{\beta}}\lambda^{m+1}}\right. \tag{3.12}$$
$$\left.-\alpha\sum_{f=0}^{\infty}\binom{\alpha v+\alpha-1}{f}(1+\lambda)^{\alpha v+\alpha-f-1}\frac{\Gamma\left(\frac{m+d+\beta}{\beta}\right)}{\beta(v+1)^{\frac{f+m+1}{\beta}}\lambda^{m+1}}\right].$$

4. Mean Deviations, Lorenz and Bonferroni Curves

In this section, we present the mean deviation about the mean, the mean deviation about the median, Lorenz and Bonferroni curves. Bonferroni and Lorenz curves are income inequality measures that are also useful and applicable to other areas including reliability, demography, medicine and insurance. The mean deviation about the mean and mean deviation about the median are defined by

$$D(\mu)=\int_0^{\infty}|x-\mu|\,g(x)dx,\quad D(M)=\int_0^{\infty}|x-M|\,g(x)dx, \tag{4.1}$$

respectively, where $\mu=E(X)$ and $M=Median(X)=G^{-1}(1/2)$ is the median of G.

These measures $D(\mu)$ and $D(M)$ can be calculated using the relationships:

$$D(\mu)=2\mu G(\mu)-2\mu+2\int_{\mu}^{\infty}xg(x)dx=2\mu G(\mu)-2\int_0^{\mu}xg(x)dx, \tag{4.2}$$

and

$$D(M)=-\mu+2\int_M^{\infty}xg(x)dx=\mu-2\int_0^{M}xg(x)dx. \tag{4.3}$$

In order to calculate these, we consider the following lemma:

Lemma 1

Let

$$
\begin{aligned}
L_1(\lambda,\theta,\alpha,\beta,a,t) = {} & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j}{k}\frac{(-1)^k\theta^{j+1}\lambda}{(1+\lambda)^{\alpha k+\alpha}(e^{\theta}-1)j!} \\
& \times\Bigg[\beta\sum_{p=0}^{\infty}\binom{\alpha k+\alpha}{p}(1+\lambda)^{\alpha k+\alpha-p}\lambda^{p+\beta-1}\int_0^t x^{p+\beta-1+a}e^{-(k+1)(\lambda x)^{\beta}}dx \\
& -\alpha\sum_{p=0}^{\infty}\binom{\alpha k+\alpha-1}{p}(1+\lambda)^{\alpha k+\alpha-q-1}\lambda^{q}\int_0^t x^{q+a}e^{-(k+1)(\lambda x)^{\beta}}dx\Bigg].
\end{aligned}
$$

then,

$$
\begin{aligned}
L_1(\lambda,\theta,\alpha,\beta,a,t) = {} & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j}{k}\frac{(-1)^k\theta^{j+1}\lambda}{(1+\lambda)^{\alpha k+\alpha}(e^{\theta}-1)j!} \\
& \times\Bigg[\beta\sum_{p=0}^{\infty}\binom{\alpha k+\alpha}{p}(1+\lambda)^{\alpha k+\alpha-p}\lambda^{p+\beta-1}\frac{\gamma((p+\beta+a)/\beta,(k+1)(tx)^{\beta})}{\beta\lambda^{p+\beta+a}(k+1)^{(p+\beta+a)/\beta}} \\
& -\alpha\sum_{q=0}^{\infty}\binom{\alpha k+\alpha-1}{q}(1+\lambda)^{\alpha k+\alpha-q-1}\lambda^{q}\frac{\gamma((q+a+1)/\beta,(k+1)(tx)^{\beta})}{\beta\lambda^{q+a+1}(k+1)^{(q+a+1)/\beta}}\Bigg].
\end{aligned}
\tag{4.4}
$$

Proof. Consider,

$$
\int_0^t x^{p+\beta-1+a}e^{-(k+1)(\lambda x)^{\beta}}dx.
$$

Let $u=(k+1)(\lambda x)^{\beta}$, then $\frac{du}{dx}=\beta(k+1)\lambda^{\beta}x^{\beta-1}$ and $x=\left[\frac{u}{(k+1)\lambda^{\beta}}\right]^{1/\beta}$.

The above integral can be rewritten by using the lower incomplete gamma function $\gamma(b,s)=\int_0^s y^{b-1}e^{-y}dy$ as,

$$
\int_0^{(k+1)(tx)^{\beta}}\frac{u^{(p+a)/\beta}}{\beta\lambda^{p+\beta+a}(k+1)^{(p+\beta+a)/\beta}}du=\frac{\gamma((p+\beta+a)/\beta,(k+1)(tx)^{\beta})}{\beta\lambda^{p+\beta+a}(k+1)^{(p+\beta+a)/\beta}}
$$

Now consider,

$$
\int_0^t x^{q+a}e^{-(k+1)(\lambda x)^{\beta}}dx.
$$

Again the above integral can be rewritten as,

$$\int_0^{(k+1)(tx)^\beta} \frac{u^{(q+a+1)/\beta}}{\beta\lambda^{q+a+1}(k+1)^{(q+a+1)/\beta}}du = \frac{\gamma((q+a+1)/\beta,(k+1)(tx)^\beta)}{\beta\lambda^{q+a+1}(k+1)^{(q+a+1)/\beta}}$$

Consequently,

$$\begin{aligned} L_1(\lambda,\theta,\alpha,\beta,a,t) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{j}{k}\frac{(-1)^k\theta^{j+1}\lambda}{(1+\lambda)^{\alpha k+\alpha}(e^\theta-1)j!} \\ &\times\left[\beta\sum_{p=0}^{\infty}\binom{\alpha k+\alpha}{p}(1+\lambda)^{\alpha k+\alpha-p}\lambda^{p+\beta-1}\frac{\gamma((p+\beta+a)/\beta,(k+1)(tx)^\beta)}{\beta\lambda^{p+\beta+a}(k+1)^{(p+\beta+a)/\beta}}\right. \\ &\left.-\alpha\sum_{q=0}^{\infty}\binom{\alpha k+\alpha-1}{q}(1+\lambda)^{\alpha k+\alpha-q-1}\lambda^{q}\frac{\gamma((q+a+1)/\beta,(k+1)(tx)^\beta)}{\beta\lambda^{q+a+1}(k+1)^{(q+a+1)/\beta}}\right]. \end{aligned}$$

Using Lemma 1, we have

$$D(\mu) = 2\mu G(\mu) - 2L_1(\lambda,\theta,\alpha,\beta,1,\mu), \tag{4.4}$$

and

$$D(M) = \mu - 2L_1(\lambda,\theta,\alpha,\beta,1,M). \tag{4.5}$$

Lorenz and Bonferroni curves are given by

$$L(G(x)) = \frac{\int_0^x tg(t)dt}{E(X)}, \quad \text{and} \quad B(G(x)) = \frac{L(G(x))}{G(x)}, \tag{4.7}$$

or

$$L(c) = \frac{1}{\mu}\int_0^d tg(t)dt, \quad \text{and} \quad B(c) = \frac{1}{c\mu}\int_0^d tg(t)dt, \tag{4.8}$$

respectively, where $d = G^{-1}(c)$. Applying Lemma 1, we can re-write the Lorenz and Bonferroni curves as

$$B(c) = \frac{1}{c\mu}\int_0^d tg(t)dt = \frac{1}{c\mu}\int_0^d xg(x)dx$$
$$B(C) = \frac{1}{c\mu}L_1(\lambda,\theta,\alpha,\beta,1,d).$$

and

$$L(c) = \frac{1}{\mu}\int_0^d tg(t)dt = \frac{1}{\mu}\int_0^d xg(x)dx$$

$$L(C) = \frac{1}{\mu}L_1(\lambda,\theta,\alpha,\beta,1,d).$$

5. Some Measures of Uncertainty

In this section, we present Shannon entropy [21], [22], as well as the Renyi entropy [19] for the EPL distribution. The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

5.1 Shannon Entropy

Shannon entropy is defined to be

$$H\left[g(X;\alpha,\beta,\theta,\lambda)\right] = E\left[-\log(g(X;\alpha,\beta,\theta,\lambda))\right]. \tag{5.1}$$

Taking the negative logarithm of the ELP pdf gives

$$\begin{aligned} -\log[g(x)] = \log\left[\frac{(e^{\theta}-1)(1+\lambda)^{\alpha}}{\theta\lambda}\right] - (\alpha-1)\log(1+\lambda+\lambda x) + (\lambda x)^{\beta} \\ -\log\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right] \\ -\theta\left[1-\left(\frac{1+\lambda+\lambda x}{1+\lambda}\right)^{\alpha} e^{-(\lambda x)^{\beta}}\right]. \end{aligned} \tag{5.2}$$

Note that, for $|x|<1$, using the identity

$$\log(1+x) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{k}, \tag{5.3}$$

we have,

$$\begin{aligned} \log[1+\lambda+\lambda x] &= \sum_{\omega=1}^{\infty}\frac{(-1)^{\omega+1}[\lambda(1+x)]^{\omega}}{\omega} \\ &= \sum_{\omega=1}^{\infty}\sum_{m=0}^{\infty}\binom{\omega}{m}\frac{(-1)^{\omega+1}\lambda^{\omega}x^m}{\omega}. \end{aligned} \tag{5.4}$$

Also, using the above identity we have

$$
\begin{aligned}
&\log\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right] \\
&=\log\left[1+\left\{\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-(1+\alpha)\right\}\right] \\
&=\sum_{a=1}^{\infty}\frac{(-1)^{a+1}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-(1+\alpha)\right]^{a}}{a} \\
&=\sum_{a=1}^{\infty}\sum_{b=0}^{\infty}\binom{a}{b}\frac{(-1)^{1-b}(1+a)^{a-b}}{a}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}\right]^{b} \\
&=\sum_{a=1}^{\infty}\sum_{b=0}^{\infty}\binom{a}{b}\frac{(-1)^{1-b}(1+a)^{a-b}\beta^{b}(\lambda x)^{b(\beta-1)}}{a}\sum_{h=0}^{\infty}\binom{b}{h}(1+\lambda)^{b-h}(\lambda x)^{h} \\
&=\sum_{a=1}^{\infty}\sum_{b=0}^{\infty}\sum_{h=0}^{\infty}\binom{a}{b}\binom{b}{h}\frac{(-1)^{1-b}(1+a)^{a-b}\beta^{b}\lambda^{b(\beta-1)+h}(1+\lambda)^{b-h}}{a}\times x^{b(\beta-1)+h}
\end{aligned}
\tag{5.5}
$$

Notice that

$$
\begin{aligned}
\left(\frac{1+\lambda+\lambda x}{1+\lambda}\right)^{\alpha}e^{-(\lambda x)^{\beta}} &=\sum_{c=0}^{\infty}\frac{(-1)^{c}(\lambda x)^{c\beta}(1+\lambda+\lambda x)^{\alpha}}{c!(1+\lambda)^{\alpha}} \\
&=\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{\alpha}{d}\frac{(-1)^{c}\lambda^{c\beta+d}x^{c\beta+d}}{c!(1+\lambda)^{d}}
\end{aligned}
\tag{5.6}
$$

By using equation (5.1), the Shannon entropy is given by

$$
\begin{aligned}
H[g(X)] &= E[-\log(g(X))] \\
&=\log\left[\frac{(e^{\theta}-1)(1+\lambda)^{\alpha}}{\theta\lambda}\right]-(\alpha-1)\sum_{\omega=1}^{\infty}\sum_{m=0}^{\infty}\binom{\omega}{m}\frac{(-1)^{\omega+1}\lambda^{\omega}E(X^{m})}{\omega} \\
&+\lambda E(X^{\beta})-\sum_{a=1}^{\infty}\sum_{b=0}^{\infty}\sum_{h=0}^{\infty}\binom{a}{b}\binom{b}{h}\frac{(-1)^{1-b}(1+\alpha)^{a-b}\beta^{b}\lambda^{b(\beta-1)+h}(1+\lambda)^{b-h}}{a} \\
&\times E\left[X^{b(\beta-1)+h}\right]-\theta\left\{1-\sum_{c=0}^{\infty}\sum_{d=0}^{\infty}\binom{\varepsilon}{d}\frac{(-1)^{c}\lambda^{c\beta+d}E[X^{c\beta+d}]}{c!(1+\lambda)^{d}}\right\}.
\end{aligned}
\tag{5.7}
$$

All the expected values can be obtained by substituting $r=m,\beta,b(\beta-1)+h,$ and $c\beta+d$ into equation (3.4) which gives the r^{th} moment.

5.2 Renyi Entropy

Renyi entropy is an extension of Shannon entropy. Renyi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty [g_{ELP}(x;\alpha,\beta,\theta,\lambda)]^v dx\right),\quad v\neq 1, v>0. \tag{5.8}$$

Renyi entropy tends to Shannon entropy as $v\to 1$. Notice that

$$\begin{aligned}[g(x)]^v &= \left[\frac{\theta\lambda}{(1+\lambda)^\alpha(e^\theta-1)}\right]^v (1+\lambda+\lambda x)^{\alpha v-v}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]^v \\ &\quad \times e^{-v(\lambda x)^\beta}\exp\left\{\theta v\left[1-\left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^\alpha e^{-(\lambda x)^\beta}\right]\right\}.\end{aligned} \tag{5.9}$$

Consider

$$\begin{aligned}&(1+\lambda+\lambda x)^{\alpha v-v}\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]^v \\ &\times e^{-v(\lambda x)^\beta}\exp\left\{\theta v\left[1-\left(\frac{1+\lambda+\lambda x}{\lambda+1}\right)^\alpha e^{-v(\lambda x)^\beta}\right]\right\} \\ &=\sum_{j=0}^\infty\sum_{k=0}^\infty\binom{j}{k}\frac{(-1)^k(\theta v)^j}{j!(1+\lambda)^{\alpha k}}(1+\lambda+\lambda x)^{\alpha k+\alpha v-v} \\ &\times\left[\beta(1+\lambda+\lambda x)(\lambda x)^{\beta-1}-\alpha\right]^v e^{-(v+j)(\lambda x)^\beta} \\ &=\sum_{j,k=0}^\infty\sum_{m=0}^\infty\binom{j}{k}\binom{v}{m}\frac{(-1)^k(\theta v)^j(-\alpha)^{v-m}}{j!(1+\lambda)^{\alpha k}} \\ &\times\beta^m(\lambda x)^{m(\beta-1)}\left[1+\lambda+\lambda x\right]^{\alpha k+\alpha v+m-v}e^{-(v+j)(\lambda x)^\beta} \\ &=\sum_{j,k=0}^\infty\sum_{m=0}^\infty\sum_{w=0}^\infty\binom{j}{k}\binom{v}{m}\binom{\alpha k+\alpha v+m-v}{w}\frac{(-1)^k(\theta v)^j(-\alpha)^{v-m}}{j!(1+\lambda)^{\alpha k}} \\ &\times(1+\lambda)^{\alpha k+\alpha v+m-v-w}\beta^m(\lambda x)^{m(\beta-1)+w}e^{-(v+j)(\lambda x)^\beta}\end{aligned} \tag{5.10}$$

Let $u=(v+j)(\lambda x)^\beta$, then $du=\beta(v+j)\lambda^\beta x^{\beta-1}dx$ and $x=\dfrac{u^{\frac{1}{\beta}}}{\lambda(v+j)^{\frac{1}{\beta}}}$.

We have

$$\int_0^\infty x^{m\beta-m+w} e^{-(v+j)(\lambda x)^\beta} dx = \frac{1}{\beta(v+j)\lambda^\beta}\int_0^\infty x^{m\beta-\beta-m+w+1}$$
$$\times e^{-(v+j)(\lambda x)^\beta}\beta(v+j)\lambda^\beta dx$$
$$= \frac{1}{\beta\lambda^{m\beta+w-m+1}(v+j)^{\frac{m\beta+w-m+1}{\beta}}} \tag{5.11}$$
$$\times\int_0^\infty u^{\frac{m\beta+w-m+1}{\beta}-1} e^{-u} du$$
$$= \frac{\Gamma(\frac{m\beta+w-m+1}{\beta})}{\beta\lambda^{m\beta+w-m+1}(v+j)^{\frac{m\beta+w-m+1}{\beta}}}$$

Thus the Renyi entropy is given by

$$I_R(v) = \frac{1}{1-v}\log\left(\sum_{j.k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{w=0}^{\infty}\binom{j}{k}\binom{v}{m}\binom{\alpha k+\alpha v+m-v}{w}\right.$$
$$\times\frac{(-1)^{k+v-m}(\theta v)^j(\alpha)^{v-m}}{j!}$$
$$\times\frac{\beta^{m-1}(1+\lambda)^{\alpha v+m-v-w}\Gamma\left(\frac{m\beta+w-m+1}{\beta}\right)}{\lambda^{w+1}(v+j)^{\frac{m\beta+w-m+1}{\beta}}} \tag{5.12}$$
$$\left.\times\left[\frac{\theta\lambda}{(1+\lambda)^\alpha(e^\theta-1)}\right]^v\right), \quad v\neq 1,\ v>0.$$

6. Maximum Likelihood Estimation

In this section, the maximum likelihood estimates of the ELP parameters λ,θ,α and β are presented. The log-likelihood, L, from the ELP distribution is given by

$$L = n\log(\theta)+n\log(\lambda)-\alpha n\log(1+\lambda)-n\log(e^\theta-1)$$
$$+(\alpha-1)\sum_{i=1}^{n}\log(1+\lambda+\lambda x_i)+\sum_{i=1}^{n}\log\left[\beta(1+\lambda+\lambda x_i)(\lambda x_i)^{\beta-1}-\alpha\right] \tag{6.1}$$
$$-\sum_{i=1}^{n}(\lambda x_i)^\beta+\theta\sum_{i=1}^{n}\left[1-\left(1+\frac{\lambda x_i}{\lambda+1}\right)^\alpha e^{-(\lambda x_i)^\beta}\right].$$

The partial derivatives of L with respect to the parameters λ,θ,α and β are:

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-\frac{ne^{\theta}}{e^{\theta}-1}+\sum_{i=1}^{n}\left[1-\left(1+\frac{\lambda x_i}{\lambda+1}\right)^{\alpha}e^{-(\lambda x_i)^{\beta}}\right], \tag{6.2}$$

$$\begin{aligned}\frac{\partial L}{\partial \alpha}&=-n\log(1+\lambda)+\sum_{i=1}^{n}\log(1+\lambda+\lambda x_i)\\&-\sum_{i=1}^{n}\log\left[\beta(1+\lambda+\lambda x_i)(\lambda x_i)^{\beta-1}-\alpha\right]^{-1}\\&-\theta\sum_{i=1}^{n}e^{-(\lambda x_i)^{\beta}}\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right)^{\alpha}\log\left(\frac{1+\lambda+\lambda x_i}{1+\lambda}\right),\end{aligned} \tag{6.3}$$

$$\begin{aligned}\frac{\partial L}{\partial \beta}&=\sum_{i=1}^{n}\frac{(1+\lambda+\lambda x_i)(\lambda x_i)^{\beta-1}\left[\beta\log(\lambda x_i)+1\right]}{\beta(1+\lambda+\lambda x_i)(\lambda x_i)^{\beta-1}-\alpha}\\&-\sum_{i=1}^{n}(\lambda x_i)^{\beta}\log(\lambda x_i)+\theta\sum_{i=1}^{n}\left(1+\frac{\lambda x_i}{\lambda+1}\right)^{\alpha}(\lambda x_i)^{\beta}e^{-(\lambda x_i)^{\beta}}\log(\lambda x_i),\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}\frac{\partial L}{\partial \lambda}&=\frac{n}{\lambda}-\frac{\alpha n}{1+\lambda}+(\alpha-1)\sum_{i=1}^{n}\frac{1+x_i}{1+\lambda+\lambda x_i}-\beta\lambda^{\beta-1}\sum_{i=1}^{n}x_i^{\beta}\\&+\sum_{i=1}^{n}\frac{\beta\left[\beta(1+\lambda+\lambda x_i)-x_i\right](\lambda x_i)^{\beta-2}}{\beta(1+\lambda+\lambda x_i)(\lambda x_i)^{\beta-1}-\alpha}\frac{\theta}{(1+\lambda)^{\alpha}}\\&\times\sum_{i=1}^{n}x_i(1+\lambda+\lambda x_i)^{\alpha-1}e^{-(\lambda x_i)^{\beta}}\left[\beta(1+\lambda+\lambda x_i)(\lambda x_i)^{\beta-1}-\frac{\alpha}{1+\lambda}\right].\end{aligned} \tag{6.5}$$

The maximum likelihood estimates, $\hat{\Delta}$ of $\Delta=(\alpha,\beta,\lambda,\theta)$ are obtained by solving the nonlinear equations $\frac{\partial \ell}{\partial \alpha}=0,$ $\frac{\partial \ell}{\partial \beta}=0,$ $\frac{\partial \ell}{\partial \lambda}=0$ and $\frac{\partial \ell}{\partial \theta}=0,$ where $\ell=\sum_{i=1}^{n}L_i.$ These equations are not in closed form and must be solved via iterative methods such as Newton-Raphson method.

6.1 Fisher Information Matrix

A measure of the amount of information is presented in this section. This is the information that cab be used when obtaining bounds on the variance of estimators and as well as approximate the sampling distribution of an estimator obtained from a large sample. We can also use it to obtain an approximate confidence interval in the case of large sample.

Let X be a random variable with the ELP pdf $f_{ELP}(\cdot;\Delta)$, where $\Delta=(\delta_1,\delta_2,\delta_3,\delta_4)^T=(\alpha,\beta,\lambda,\theta)^T$. Then, Fisher information matrix (FIM) is the 4×4 symmetric matrix with elements:

$$\mathbf{I}_{ij}(\Delta) = E_{\Delta}\left[\frac{\partial \log\left(f_{ELP}\left(X;\Delta\right)\right)}{\partial \delta_i}\frac{\partial \log\left(f_{ELP}\left(X;\Delta\right)\right)}{\partial \delta_j}\right]$$

If the density $f_{ELP}(\cdot;\Delta)$ has a second derivative of all i and j, then an alternative expression for $\mathbf{I}_{ij}(\Delta)$ is

$$\mathbf{I}_{ij}(\Delta) = E_{\Delta}\left[\frac{\partial^2 \log\left(f_{ELP}\left(X;\Delta\right)\right)}{\partial \delta_i \partial \delta_j}\right]$$

Since all second derivatives exist for the ELP; the formula above is therefore appropriate and significantly simplifies the computations. Elements of the FIM can be numerically obtained by MATLAB or MAPLE software. The total FIM $\mathbf{I}_n(\Delta)$ can be approximated by

$$\mathbf{J}_n(\hat{\Delta}) = \left[-\frac{\partial^2 \log L}{\partial \delta_i \partial \delta_j}\bigg|_{\Delta=\hat{\Delta}} \right]_{4\times 4} \tag{6.6}$$

For real data, the matrix given in Equation (6.6) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software.

6.2 Asymptotic Confidence Intervals

In this section, we present the asymptotic confidence intervals for the parameters of the ELP distribution. Let $\hat{\Delta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$ be the maximum likelihood estimate of $\Delta = (\alpha, \beta, \lambda, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\Delta} - \Delta) \xrightarrow{d} N_4(\underline{0}, I^{-1}(\Delta))$, where $I(\Delta)$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\Delta)$ is replaced by the observed information matrix evaluated at $\hat{\Delta}$, that is $J(\hat{\Delta})$. The multivariate normal distribution $N_4(\underline{0}, J(\hat{\Delta})^{-1})$, where the mean vector $\underline{0} = (0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for α, β, λ and θ are given by:

$$\hat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\alpha\alpha}(\hat{\Delta})}, \quad \hat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\beta\beta}(\hat{\Delta})}, \quad \hat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\lambda\lambda}(\hat{\Delta})}, \quad \text{and} \quad \hat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{I^{-1}_{\nu\theta}(\hat{\Delta})},$$

respectively, where $I_{\alpha\alpha}^{-1}(\Delta)$, $I_{\beta\beta}^{-1}(\Delta)$, $I_{\lambda\lambda}^{-1}(\Delta)$, and $I_{\theta\theta}^{-1}(\Delta)$ are the diagonal elements of $I_n^{-1}(\Delta)$, and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

The likelihood ratio (LR) test can be used to compare the fit of the ELP distribution with its sub-models for a given data set. In fact, to test $\alpha=\beta=1$, the LR test statistic $\Phi=2[\ln L(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})-\ln L(1,1,\tilde{\lambda},\tilde{\theta})]$, where $\hat{\alpha},\hat{\beta},\hat{\lambda},$ and $\hat{\theta}$ are the unrestricted estimates, whereas $\tilde{\lambda}$ and $\tilde{\theta}$ are the restricted estimates is used. This can be used to compare the EL distribution to the ELP distribution. The LR test rejects the null hypothesis H_0 if $\lambda>\chi_\eta^2$, where χ_η^2 denotes the upper $100\eta\%$ point of the χ^2 distribution with 1 degree of freedom.

7. Simulation Study

In this section, we study the performance and accuracy of maximum likelihood estimates of the ELP model parameters by conducting various simulations for different sample sizes and different parameter values. Equation (2.3) is used to generate random data from the ELP distribution. The simulation study is repeated for $N=5,000$ times each with sample size $n=25,\ 50,\ 150,\ 300,\ 500$ and parameter values I : $\alpha=0.5,\ \ \beta=1.0,\ \theta=0.5,\ \lambda=1.0$ and II : $\alpha=-0.5,\ \beta=1.0,\ \theta=0.5,\ \lambda=0.3$. Three quantities are computed in this simulation study.

(a) Average bias of the MLE $\hat{\vartheta}$ of the parameter $\vartheta=\alpha,\beta,\theta,\lambda$:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta}-\vartheta).$$

(b) Root mean squared error (RMSE) of the MLE $\hat{\vartheta}$ of the parameter $\vartheta=\alpha,\beta,\theta,\lambda$:

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(\hat{\vartheta}-\vartheta)^2}.$$

(c) Average width (AW) of 95% confidence intervals of the parameter $\vartheta=\alpha,\beta,\theta,\lambda$.

Table (7.1) presents the Average Bias, RMSE and AW values of the parameters α,β,θ and λ for different sample sizes. From the results, we can verify that as the sample size n increases, the RMSEs decay toward zero. We also observe that for all the parametric values, the biases decrease as the sample size n increases. Also, the table shows that the average confidence widths decrease as the sample size increases.

Table 7.1. Monte Carlo Simulation Results: Average Bias, RMSE and AW

		I			II		
Parameter	n	Average Bias	RMSE	AW	Average Bias	RMSE	AW
α	25	-0.57384	2.18048	23.96995	-0.54284	2.21185	15.79877
	50	-0.36661	1.94933	17.28701	-0.49208	1.99506	12.79267
	150	-0.28249	1.87608	11.04948	-0.11091	1.86883	10.51284
	300	-0.23816	1.71440	8.18401	-0.08676	1.78080	9.31420
	500	-0.18954	1.35800	6.28867	-0.03757	1.62068	7.84569
β	25	0.64174	4.10281	4.15241	0.57529	4.24467	4.11420
	50	0.10856	1.87325	1.93061	0.42333	2.37828	2.18112
	150	-0.07511	0.37447	0.91812	-0.11326	0.69649	1.21709
	300	-0.06785	0.15846	0.69016	-0.10684	0.26227	0.89936
	500	-0.05072	0.11324	0.54766	-0.08277	0.16863	0.75821
θ	25	1.10609	2.04620	11.64127	2.73492	17.45171	34.44374
	50	1.05417	1.76991	9.03825	1.61995	4.00652	12.65466
	150	0.85854	1.34670	6.90524	1.09477	1.88143	7.82394
	300	0.69699	1.18122	5.70376	0.81511	1.48771	5.88105
	500	0.55262	1.00443	4.91740	0.62459	1.20327	4.65968
λ	25	0.46270	3.63233	18.69358	4.18358	81.68417	109.27790
	50	0.43782	1.33186	10.02093	0.80594	8.35183	9.61811
	150	0.37098	0.73324	5.51393	0.37230	0.84799	3.26208
	300	0.29883	0.65441	3.81555	0.24081	0.61491	1.97089
	500	0.21108	0.53428	2.76988	0.15427	0.48389	1.29659

8. Application

In this section, we demonstrate the usefulness of the ELP model by fitting some real data set. We fit the density functions of the extended Lindley (EL) [2], exponentiated Weibull Poisson (EWP) [12], Weibull-Poisson (WP) [14], exponential Poisson (EP) [23] and Kumaraswamy Weibull (KW) [6] distributions. Notice that the WP and EP distributions are submodels of the ELP distribution. The density functions of the EWP and KW distributions are respectively given by

$$f_{EWP}(x)=\frac{\alpha\lambda\theta\beta^{\alpha}x^{\alpha-1}}{e^{\theta}-1}e^{-(\beta x)^{\alpha}}\left[1-e^{-(\beta x)^{\alpha}}\right]^{\lambda-1}\exp\left\{\theta\left[1-e^{-(\beta x)^{\alpha}}\right]^{\lambda}\right\}, \tag{8.1}$$

where $\alpha,\beta,\lambda,\theta>0,$

and

$$f_{KW}(x) = ab\alpha\theta x^{\theta-1}e^{-\alpha x^{\theta}}\left[1-e^{-\alpha x^{\theta}}\right]^{a-1}\left[1-\left(1-e^{-\alpha x^{\theta}}\right)^{a}\right]^{b-1}, \tag{8.2}$$

for $x > 0, \alpha > 0, \theta > 0, a > 0, b > 0.$

Estimates of the parameters of the distributions, standard errors (in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC) and Bayesian Information Criterion ($BIC = p\log(n) - 2\log(\hat{L})$), where $\hat{L} = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are obtained. The goodness-of-fit statistics: Cramer von Mises (W^*), Anderson-Darling (A^*), which are described in detail by Chen and Balakrishnan [4] and sum of squares (SS) from the probability plots for the data sets are presented.

Plots of the fitted densities, the histogram of these data and probability plots according to Chambers [3] are presented in Figure 9.1 and Figure 9.2. For the probability plot, we plotted $G_{ELP}(x_{(j)};\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$ against $\frac{j-0.375}{n+0.25}$, $j = 1, 2, \cdots, n,$ where $x_{(j)}$ are the ordered values of the observed data.

The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in [13]. The data is given in the table below.

Table 8.1. Bladder Cancer Patients Data

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23	3.52	4.98	6.97
9.02	13.29	0.40	2.26	3.57	5.06	7.09	9.22	13.80	25.74	0.50	2.46	3.64
5.09	7.26	9.47	14.24	25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31
0.81	2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32	7.39	10.34
14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05	2.69	4.23
5.41	7.62	10.75	16.62	43.01	1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26
2.83	4.33	5.49	7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85	8.26	11.98	19.13
1.76	3.25	4.50	6.25	8.37	12.02	2.02	3.31	4.51	6.54	8.53	12.03	20.28
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65	12.63	22.69		

Table 8.2. Estimates From Bladder Cancer Patients Data

	Estimates				Statistics						
Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$-2\log L$	AIC	AICC	BIC	SS	A*	W*
ELP	-1.2482	0.5268	0.2020	5.2553	818.9	826.9	827.3	838.3	0.01217	0.0839	0.0134
	(1.6972)	(0.1801)	(0.2442)	(1.8322)							
WP	0	0.5098	0.8320	7.1295	819.6	825.6	825.8	834.1	0.01921	0.1424	0.0221
	-	(0.07813)	(0.4503)	(2.2962)							
EP	0	1	0.1113	0.1603	828.6	832.6	832.7	838.3	0.01921	0.7669	0.1283
	-		(0.0226)	(0.7187)							
EL	-2.0287	1.2242	0.0445	-	827.1	833.1	833.3	841.7	0.17540	0.5241	0.0863
	(4.0477)	(0.2637)	(0.0569)	-							
EWP	0.4307	1.9961	3.4083	4.1423	818.9	826.9	827.2	838.3	0.01489	0.1166	0.0178
	(0.1060)	(2.4950)	(3.4097)	(2.2430)							
	α	θ	a	b							
KW	0.5305	0.4011	4.7755	4.2949	821.2	829.2	829.5	840.6	0.03398	0.2766	0.0422
	(0.8797)	(0.7793)	(11.5400)	(20.0657)							

For the bladder cancer patients data, the LR test statistic for the hypothesis H_0: $WP(0,\beta,\lambda,\theta)$ against H_a: $ELP(\alpha,\beta,\lambda,\theta)$, is $\Phi = 819.6 - 818.9 = 0.7$. The p-value is $0.4028 > 0.05$. We therefore fail to reject H_0 and conclude that there is no significant difference between ELP and WP distributions for this data set. When $\alpha = 0$ and $\beta = 0$, ELP distribution becomes EP distribution. We can test H_0: $EP(0,1,\lambda,\theta)$ against H_a : $ELP(\alpha,\beta,\lambda,\theta)$, to obtain $\Phi = 828.6 - 818.9 = 9.7$. The p-value is $7.83\times10^{-3} < 0.05$. We reject the null hypothesis of EP distribution and conclude that the ELP distribution is significantly better than the EP distribution. When the ELP distribution is compared to other 4 parameter models such as the EWP and KW, it proves to be superior to both of them based on the SS, A^* and W^* values. The ELP yields the smallest values among the 3 models with 4 parameters. The probability plot shows the ELP distributions is a better fit as compared to all the other models.

The asymptotic covariance matrix of the MLEs of the ELP model parameters, which is the inverse of the observed Fisher information matrix, **FIM**, $\mathbf{I}_n^{-1}(\hat{\Delta})$, is given by

$$\begin{pmatrix} 2.8806 & 0.1966 & 0.3974 & 1.0635 \\ 0.1966 & 0.0324 & 0.0186 & -0.1334 \\ 0.3974 & 0.0186 & 0.0596 & 0.2609 \\ 1.0635 & -0.1334 & 0.2609 & 3.3568 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for α, β, λ and θ are given by $-1.2482 \pm 3.3266, 0.5268 \pm .3528, 0.2020 \pm 0.4785,$ and 5.2553 ± 3.5910, respectively. Plots of the fitted densities and histogram, observed probability versus predicted probability for the bladder cancer patients data are shown in the figures below.

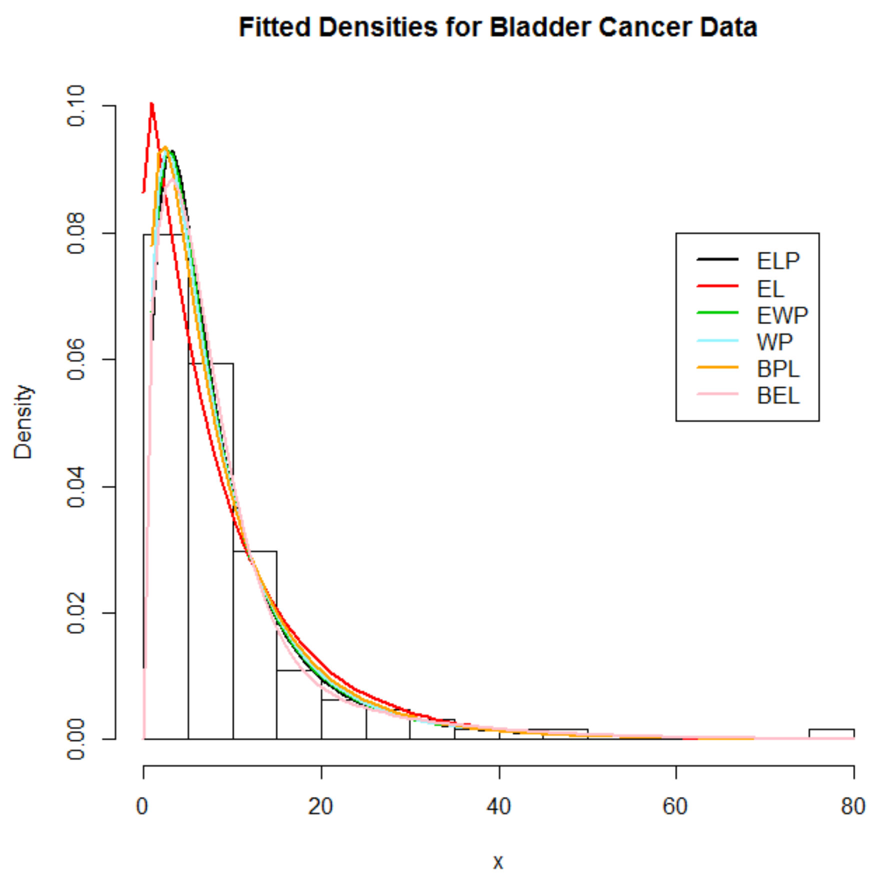

Table 8.3. Estimates From Bladder Cancer Patients Data

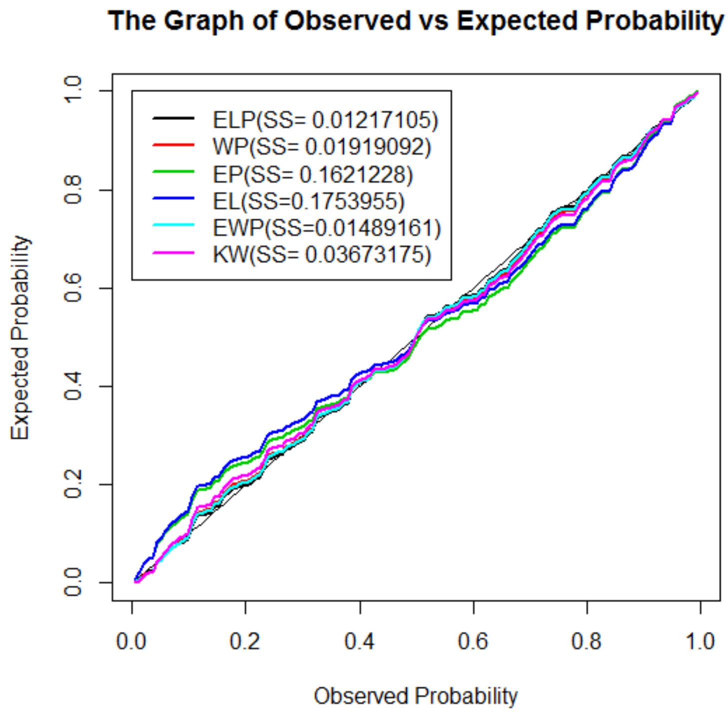

Figure 8.2. Graph of Observed vs Expected Probability for Bladder Cancer Data

9. Concluding Remarks

We have proposed and presented results on a new class of distributions called the extended Lindley Poisson distribution which is a generalization of the extended Lindley distribution. This class of distributions has applications in lifetime data analysis. Properties of this class of distributions including moments, hazard function, reverse hazard function, quantile function, income inequality measures such as Lorenz and Bonferroni curves are derived. Renyi entropy, order statistics, mean and median deviations are presented. Estimation of the parameters of the models and an application are also given.

References

[1]. Asgharzedah, H.S. Bakouch and H. Esmaeli, Pareto Poisson-Lindley Distribution with Applications, Journal of Applied Statistics, 40 (8) (2013).

[2]. H.S. Bakouch,, B.M. Al-Zahrani, A.A. Al-Shomrani, V.A.A. Marchi and F. Louzada, An extended Lindley distribution, Journal of the Korean Statistical Society, 41, 75-85, 2012.

[3]. Chambers, J., Cleveland, W., Kleiner, B., and Tukey, P., Graphical Methods of Data Analysis, Chapman and

Hall, 1983.

[4]. Chen, G. and Balakrishnan, N., A General Purpose Approximate Goodness-of-fitt Test, Journal of Quality Technology, 27, 154-161, (1995).

[5]. Cohen, A. C. (1960). Estimating the parameter in a conditional Poisson distribution. Biometrics, 16 (2), 203-211.

[6]. Cordeiro, G.M., Ortega, E.M.M. and Nadarajah, S., The Kumaraswamy Weibull Distribution with Application to Failure Data, Journal of Franklin Institute, 347, 1399-1429, (2010).

[7]. R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey and D.E. Knuth, On the Lambert W function. Advances in Computational Mathematics, 5 (1996), 329-359.

[8]. M.E. Ghitany, B. Atieh and S. Nadarajah, Lindley Distribution and Its Applications, Mathematics and Computers in Simulation, 78 (4) (2008), 493-506.

[9]. M.E. Ghitany, D.K. Al-Mutairi, N. Balakrishnan and L.J. Al-Enezi, Power Lindley distribution and associated inference, Computational Statistics and Data Analysis, 64(2013), 20-33.

[10]. D.V. Lindley, Fiducial distributions and Bayes theorem, Journal of the Royal Statistical Society, Series B 20(1958), 102-107.

[11]. V.Merovci and I. Elbatai, Transmuted Lindley-Geometric Distribution and its Applications, Journal of Statistics Applications and Probability, An International Journal, 3(1), 77-91, (2014).

[12]. Mahmoudi, E. and Sepahdar, A., Exponentiated Weibull-Poisson Distribution: Model, Properties and Applications, Mathematics and Computers in Simulation, 92, 76-97 (2013).

[13]. Lee, E. T., and Wang, J. (2003). Statistical methods for survival data analysis (Vol. 476). John Wiley & Sons.

[14]. Lu, W. and Shi, D., A New Compounding Lifetime Distribution: the Weibull-Poisson Distribution, Journal of Applied Statistics, 39, 21-38, (2012).

[15]. S. Nadarajah, H.S. Bakouch and R. Tahmasbi A Generalized Lindley Distribution, Sankhya B, 73(2011), 331-359.

[16]. Oluyede, B. O., and Yang, T. (2015). A new class of generalized Lindley distributions with applications. Journal of Statistical Computation and Simulation, 85 (10), 2072-2100.

[17]. Pararai, M., Warahena-Liyanage, G. and Oluyede, B.O., Exponentiated Power Lindley Poisson Distribution: Properties and Application, (Submitted to Communications in Statistics, Theory and Methods), (2015).

[18]. Pararai, M., Warahena-Liyanage, G. and Oluyede, B.O.,A New Class of Generalized Power Lindley Distribution with Applications to Lifetime Data, Theoretical Mathematics and Applications, vol.5, no.1, 2015, 53-96, ISSN: 1792-9687 (print), 1792-9709 (online). Scienpress Ltd, 2015

[19]. A. Renyi, On measures of entropy and information, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, vol. I, University of California Press, Berkeley, (1961), 547-561.

[20]. M. Sankaran, The Discrete Poisson-Lindley Distribution, Biometrics, 26 (1) (1970), 145-149.

[21]. E.A. Shannon, A Mathematical Theory of Communication, The Bell System Technical Journal, 27(10)(1948), 379-423.

[22]. E.A. Shannon, A Mathematical Theory of Communication, The Bell System Technical Journal, 27 (10) (1948), 623-656.

[23]. Tahmasbi, R., and Rezaei, S. (2008). A two-parameter lifetime distribution with decreasing failure rate. Computational Statistics & Data Analysis, 52 (8), 3889-3901.

[24]. Warahena-Liyanage, G. and Pararai, M., The Lindley Power Series Class of Distributions:Model, Properties and Applications, Journal of Computations and Modelling, vol.5, no.3, 2015, 35-80,ISSN: 1792-7625 (print), 1792-8850 (online) Scienpress Ltd, 2015

[25]. Zakerzadeh, H., and Dolati, A. (2009). Generalized Lindley Distribution. Journal of Mathematical Extension, 3 (2), 13-25.

[26]. Zakerzadeh, H., and Mahmoudi, E. (2012). A new two parameter lifetime distribution: model and properties. arXiv preprint arXiv:1204.4248.

R algorithms

In this section, the R codes to compute cdf, pdf, moments, Renyi entropy, mean deviations, maximum likelihood estimates and variance-covariance matrix for the ELP distribution are presented.

```
#Define the pdf of ELP distribution
f1=function (x, alpha, beta, theta, lambda ){theta*lambda*
(1+lambda+lambda*x )^( alpha-1)*exp(-(lambda*x)^beta)
*exp(theta*(1-((1+lambda+lambda*x)/(1+lambda))
^alpha*exp(-(lambda*x )^beta)))*(alpha-beta*lambda^(beta-1)*x^(beta-1)
*(lambda+1)-beta*lambda^(beta )*x^beta)
/((1-exp(theta))*(1+lambda)^alpha)
y=integrate(f, lower=0, upper=Inf, subdivisions=100,
alpha=alpha, beta=beta, theta=theta, lambda=lambda)
return ( y )
}

#Define the cdf of ELP distribution
F1=function (x, alpha, beta, theta, lambda){
y=(1-exp(theta*(1-(((1+lambda+lambda*x)/
(1+lambda))^alpha)*exp(-( lambda*x)^beta))))/(1-exp(theta))
return(y)
}

#Define the moments of ELP distribution
moment=function (alpha, beta, theta, lambda, r){
f=function (x, alpha, beta, theta, lambda, r)
```

```
f(x^r)*(f1(x, alpha, beta, theta, lambda))}
y=integrate(f, lower=0, upper=Inf, subdivisions=100, alpha=alpha
, beta=beta, theta=theta, lambda=lambda, r=r)
return (y)
}

#Define the quantile of ELP distribution
quantile=function (alpha, beta, theta, lambda, u){
f=function (x)f(1-exp(theta*(1-(((1+lambda+lambda*x)
/(1+lambda))^alpha)*exp(-(lambda*x)^beta))))/(1-exp(theta))-u}
rc<-uniroot (f, lower=0, upper=100, tol=1e-9)
result=rc$root
#check
error=F1(result, alpha, beta, theta, lambda)-u
return(list("result"=result, "error"=error))
}

#Define Mean Deviation about the mean of ELP distribution
DU=function (alpha, beta, theta, lambda){
mu=moment (alpha, beta, theta, lambda, 1) $ value
f=function (x, alpha, beta, theta, lambda )
f(abs(x-mu)*f1(x, alpha, beta, theta, lambda)}
y=integrate (f, lower=0,upper=Inf, subdivisions=100
, alpha=alpha, beta=beta, theta=theta, lambda=lambda )
return (y)
}
```

```
#Define Mean Deviation about the median of ELP distribution
DM=function(alpha, beta, theta, lambda){
M=median (c(X)) #X is the data set
f=function (x, alpha, beta, theta, lambda)
f(abs(x-M)*f1(x, alpha, beta, theta, lambda)}
y=integrate (f, lower=0, upper=Inf, subdivisions=100
, alpha=alpha, beta=beta, the ta=theta, lambda=lambda)
return (y)
}

Define the Renyient ropy of ELP distribution
t=function (alpha, beta, theta, lambda, v){
f=function (x, alpha, beta, theta, lambda, v)
f(f1 (x, alpha, beta, theta, lambda))^(v)}
y=integrate (f, lower=0, upper=Inf, subdivisions=100
, alpha=alpha, beta=beta, theta=theta, lambda=lambda) $ value
return (y)
}
Renyi=function (alpha, beta, theta, lambda, v){
y=log (t(alpha, beta, theta, lambd, v))/(1-v )
return (y)
}

#Calculate the maximum likelihood estimators
#of ELP distribution
library ('bbmle')
xvec<-c(X) #X is the data set
```

```
ln<-function (alpha, beta, theta, lambd){
-sum(log(theta*lambda*(1+lambda+lambda*x )
^(alpha-1)*exp(-(lambda*x)^beta)*exp (theta*
(1-((1+lambda+lambda*x)/(1+lambda))^alpha*
exp(-(lambda*x)^beta)))*(alpha-beta*lambda^
(beta-1)*x^(beta-1)*(lambda+1)-beta*lambda^(beta)
*x^beta)/((1-exp(theta))*(1+lambda)^alpha)))
}
mle.results1<-mle2(ln, start=list (alpha=alpha, beta=beta
, theta=theta, lambda=lambda), hessian. opt=TRUE)
summary(mle. results1)

# Variance-covariance matrx of ELP distribution
vcov (mle. results1)
```