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The Derivative of a Solution to a Second Order Parameter Dependent Boundary Value Problem with a Nonlocal Integral Boundary Condition

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Abstract

We discuss derivatives of the solution of the second order parameter dependent boundary value problem with an integral boundary condition $y'' = f(x, y, y', \lambda)$, $y(x_1) = y_1$, $y(x_2) + \int_c^d ry(x)dx = y_2$ and its relationship to a second order nonhomogeneous differential equation which corresponds to the traditional variational equation. Specifically, we show that given a solution $y(x)$ of the boundary value problem, the derivative of the solution with respect to the parameter λ is itself a solution to the aforementioned nonhomogeneous equation with interesting boundary conditions.

Introduction

In this paper, we differentiate, with respect to λ, the solution of the nonlocal second order differential equation

$$y'' = f(x, y, y', \lambda), \qquad a < x < b \text{ in } \mathbb{R} \tag{1}$$

with parameter λ and boundary conditions

$$y(x_1) = y_1, \qquad y(x_2) + \int_c^d ry(x)dx = y_2, \tag{2}$$

where $a < x_1 < c < d < x_2 < b$ in $\mathbb{R}$; note the integral boundary condition. In particular, we show the

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relationship between the derivative of the solution $y(x)$ to (1), (2) and the nonhomogeneous equation along $y(x)$

$$z'' = \frac{\partial f}{\partial u_1} z + \frac{\partial f}{\partial u_2} z' + \frac{\partial f}{\partial \lambda}. \tag{3}$$

The nonhomogeneous equation (3) is a corresponding equation of the traditional variational equation along $y(x)$

$$z'' = \frac{\partial f}{\partial u_1} z + \frac{\partial f}{\partial u_2} z'. \tag{4}$$

The study of the relationship between the derivative of a solution to a differential equation and its associated variational equation is first attributed to Peano by Hartman in [5]. The theory presented in Hartman's book was developed for initial value problems. Since then, much research has been done to find analogues of Peano's work to various types of boundary value problems. One can find examples with a wide variety of domains and boundary conditions as can be seen in [3, 4, 11, 14] for differential equations, [1, 2, 7, 8, 12] for difference equations, and [13] for dynamic equations on time scales. The preceding papers were written with a parameter independent differential equation. The primary motivation for the research presented here is from two articles [10] and [6]. In the former, the authors studied differentiation of solutions of a differential equation with an integral boundary condition and parameter independence. The second article covers a discrete version of the topic but introduces the parameter λ. The work presented here expands directly upon that found in [10] by adding parameter dependence and studying the derivative of the solution with respect to theparameter. One interesting theme in each of the preceding works is the method of writing the boundary value problem in terms of an initial value problem which in turn allows the author to apply a version of Peano's theorem for initial value problems which leads to the desired result. The same technique is employed to achieve our results here.

The remainder of the paper is arranged as follows. Section 2 develops the conditions necessary for our theorems and introduces the theorems employed to obtain our results. To conclude, in Section 3 the main result of the paper is presented and followed with its proof.

Assumptions and Theorems

Before presenting the results of the paper, we will impose the following hypotheses. First, we require that the differential equation (1) and its derivatives are continuous:

(i) $f(x,u_1,u_2,\lambda):(a,b)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous,

(ii) $\partial f(x,u_1,u_2,\lambda)/\partial u_i:(a,b)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous $(i=1,2)$, and

(iii) $\partial f(x,u_1,u_2,\lambda)/\partial \lambda:(a,b)\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous.

Second, we require that solutions of (1) be unique implying the condition:

(iv) Given $r,\lambda\in\mathbb{R}$ and $a<x_1<c<d<x_2<b$ in $\mathbb{R}$, if $y(x)$ and $z(x)$ are solutions of (1), (2), then on (a,b)

$$y(x)=z(x).$$

In the same vein, we also require that solutions of the variational equation (4) are unique along solutions of (1):

(v) Given $r,\lambda\in\mathbb{R}$ and $a<x_1<c<d<x_2<b$ in $\mathbb{R}$, a solution $y(x)$ of (1), and a solution $u(x)$ of (4) along $y(x)$, if $u(x_1)=0$ and $u(x_2)+\int_c^d ru(x)dx=0$, then, on (a,b),

$$u(x)\equiv 0.$$

While the final, following condition is not required for the work, we impose it as to reduce the repetitious statement about the maximal intervals of existence of (1):

(vi) Solutions of initial value problems for (1) extend to (a,b).

As mentioned in the introduction, the main result of this work will, at its essence, be an analogue of Peano's Theorem in [5]. We state the slightly modified version of Peano's Theorem for initial value problems here. The proof follows similarly to the original and so is omitted.

Theorem 1. *Assume that, with respect to (1), conditions (i)-(iii) are satisfied. Let* $x_0,c_1,c_2\in\mathbb{R}$ *and* $y(x)\equiv y(x,x_0,c_1,c_2,\lambda)$ *denote the solution of (1) with initial conditions* $y(x_0)=c_1$ *and* $y(x_0)=c_2$.

Then,

(a) $\alpha_i(x) \equiv \partial y(x) / \partial c_i$, *for* $i = 1, 2$ *exists on* (a, b) *and is the solution of (3) with initial conditions*

$$\alpha_1(x_0) = 1, \qquad \alpha_1'(x_0) = 0,$$
$$\alpha_2(x_0) = 0, \qquad \alpha_2'(x_0) = 1.$$

(b) $L \equiv \partial y(x) / \partial \lambda$ *exists on* (a, b) *and is the solution of (3) with initial conditions*

$$L(x_0) = 0, \qquad L'(x_0) = 0.$$

The next theorem provides for the continuous dependence of the boundary data and parameter. The result utilizes the Brouwer Invariance of Domain Theorem and a typical proof may be found in [9].

Theorem 2. *Assume (i)-(iv) are satisfied with respect to (1). Let* $y(x)$ *be a solution of (1) on* (a, b), *and let* $a < \alpha < x_1 < c < d < x_2 < \beta < b$ *and* $y_1, y_2, r, \lambda \in \mathbb{R}$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $i = 1, 2$, $|x_i - t_i| < \delta$, $|c - \xi| < \delta$, $|d - \Delta| < \delta$, $|r - \rho| < \delta$, $|\lambda - \omega| < \delta$, $|u(x_1) - y_1| < \delta$, $|u(x_2) + \int_c^d ru(x)dx - y_2| < \delta$, *there exists a unique solution* $u_\delta(x)$ *of (1) with respect to the parameter* ω *such that* $u_\delta(t_1) = y_1$ *and* $u_\delta(t_2) + \int_\xi^\Delta \rho u_\delta(x)dx = y_2$, *and, for* $i = 1, 2$, $u_\delta^{(i)}$ *converges uniformly to* $u^{(i)}$ *as* $\delta o 0$ *on the interval* $[\alpha, \beta]$.

Main Result

Next, we present the main result of the paper which is our analogue of Peano's Theorem. Note that we only present the case of the derivative with respect to the parameter λ as derivatives of the boundary data were considered in [10].

Theorem 3. *Assume conditions (i)-(vi) are satisfied. Let* r, λ, y_1, y_2 *and* $a < x_1 < c < d < x_2 < b$ *in* $\mathbb{R}$. *Suppose* $y(x) \equiv y(x, x_1, x_2, y_1, y_2, c, d, r, \lambda)$ *solves (1) on* (a, b) *satisfying (2). Then,*

$$\Lambda \equiv \frac{\partial y}{\partial \lambda}$$

solves the nonhomogeneous equation (3) and satisfies the boundary conditions

$$\Lambda(x_1)=0, \qquad \Lambda(x_2)+\int_c^d r\Lambda(x)dx=0.$$

Proof. Let

$$\Lambda_h(x)=\frac{1}{h}[y(x,\lambda+h)-y(x,\lambda)]$$

define the difference quotient for the purpose of taking the derivative of y with respect to λ, and let $\delta>0$ be as in Theorem 2. For notational purposes, we suppress the boundary data as it is fixed, i.e. $y(x,\lambda)=y(x,x_1,x_2,y_1,y_2,c,d,r,\lambda)$. With that in mind, when $h\neq 0$ such that $|h|<\delta$,

$$\begin{aligned}\Lambda_h(x_1)&=\frac{1}{h}[y(x_1,\lambda+h)-y(x_1,\lambda)]\\&=\frac{1}{h}[y_1-y_1]\\&=0.\end{aligned}$$

In much the same manner, the second boundary condition is also met, for

$$\begin{aligned}\Lambda_h(x_2)+\int_c^d r\Lambda_h(x)dx&=\frac{1}{h}[y(x_2,\lambda+h)-y(x_2,\lambda)]\\&+\int_c^d\frac{r}{h}[y(x,\lambda+h)-y(x,\lambda)]dx\\&=\frac{1}{h}\{y(x_2,\lambda+h)+\int_c^d ry(x,\lambda+h)dx\\&-y(x_2,\lambda)-\int_c^d ry(x,\lambda)dx\}\\&=\frac{1}{h}\{y_2-y_2\}\\&=0.\end{aligned}$$

With each boundary condition satisfied, we now treat the boundary value problem as an initial value problem at the point x_1. To that end, let

$$\mu\equiv y'(x_1,\lambda),$$

$$\nu\equiv y'(x_1,\lambda+h)-\mu.$$

As a result, we can denote $y(x,x_1,x_2,y_1,y_2,c,d,r,\lambda)$ by $u(x,x_1,y_1,u,\lambda)$ and the difference quotient by

$$\varLambda_h(x) \equiv \frac{1}{h}[u(x,x_1,y_1,\mu+\nu,\lambda+h)-u(x,x_1,y_1,\mu,\lambda)].$$

Also, by Theorem 2, we have as $h \to 0, \nu \to 0$. Now, utilizing a telescoping sum,

$$\begin{aligned}\varLambda_h(x) &\equiv \frac{1}{h}[u(x,x_1,y_1,\mu+\nu,\lambda+h)-u(x,x_1,y_1,\mu+\nu,\lambda)\\ &+u(x,x_1,y_1,\mu+\nu,\lambda)-u(x,x_1,y_1,\mu,\lambda)]\end{aligned}$$

Applications of the Mean Value Theorem and Peano's Theorem yield

$$\begin{aligned}\varLambda_h(x) &= \frac{1}{h}[L(x,u(x,x_1,y_1,\mu+\nu,\lambda+\overline{h}))(\lambda+h-\lambda)\\ &\quad +\alpha_2(x,u(x,x_1,y_1,\mu+\overline{\nu},\lambda)(\mu+\nu-\mu)]\\ &= L(x,u(\cdot))+\frac{\nu}{h}\alpha_2(x,u(\cdot)),\end{aligned}$$

where $\lambda+\overline{h}$ is between λ and $\lambda+h$ and $\mu+\overline{\nu}$ is between μ and $\mu+\nu$. Also, we have $L(x,u(\cdot))$ and $\alpha_2(x,u(\cdot))$ as solutions to (3) with initial conditions

$$\begin{aligned} L(x_1) &= 0, & L'(x_1) &= 0,\\ \alpha_2(x_1) &= 0, & \alpha_2'(x_1) &= 1. \end{aligned}$$

Therefore, to show $\lim_{h\to 0}\varLambda_h(x)$ exists, we only need to show $\lim_{h\to 0}\nu/h$ exists. To prove this, we note that condition (v), $\alpha_2(x,u(\cdot))\neq 0$, and $\alpha_2(x_1,u(\cdot))=0$ together imply

$$\alpha_2(x_2,u(\cdot))+\int_c^d r\alpha_2(x,u(\cdot))dx \neq 0.$$

However, recall from before that

$$\varLambda_h(x_2)+\int_c^d r\varLambda_h(x)dx = 0.$$

Substituting into this equation,

$$L(x_2,u(\cdot))+\frac{\nu}{h}\alpha_2(x_2,u(\cdot))+\int_c^d r[L(x,u(\cdot))+\frac{\nu}{h}\alpha_2(x,u(\cdot))]dx=0.$$

Solving for ν / h,

$$\frac{\nu}{h}=\frac{-L(x_2,u(\cdot))-\int_c^d rL(x,u(\cdot)dx}{\alpha_2(x_2,u(\cdot))+\int_c^d r\alpha_2(x,u(\cdot))dx}.$$

Finally, we apply the limit as h approaches 0. Since the denominator of this quotient is nonzero, the limit exists, and we can define U as

$$U\equiv\lim_{h\to 0}\frac{\nu}{h}=\lim_{h\to 0}\frac{-L(x_2,u(\cdot))-\int_c^d rL(x,u(\cdot)dx}{\alpha_2(x_2,u(\cdot))+\int_c^d r\alpha_2(x,u(\cdot))dx}.$$

Thus, we have,

$$\Lambda(x)=\lim_{h\to 0}\Lambda_h(x)=L(x,u(\cdot)+U\alpha_2(x,u(\cdot))$$

which is a solution of (3). Also note, the first

$$\Lambda(x_1)=\lim_{h\to 0}\Lambda_h(x_1)=0.$$

and second

$$\Lambda(x_2)+\int_c^d r\Lambda(x)dx=\lim_{h\to 0}\{\Lambda_h(x_2)+\int_c^d r\Lambda_h(x)dx\}=0$$

boundary conditions are satisfied.

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