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Conformal Euler-Lagrangian Equations on 4-Walker Manifolds

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Abstract

The main purpose of the present paper is to study almost paracomplex structures conformal Euler-Lagrangian equations on 4-dimensional Walker manifolds. A Walker nmanifold is a semi-Riemannian n-manifold, which admits a field of parallel null r-planes, with $r \leq \frac{n}{2}$. It is well-known that semi-Riemannian geometry has an important tool to describe spacetime events. Therefore, solutions of some structures about 4-Walker manifold can be used to explain spacetime singularities. In this study, we present complex analogues of Lagrangian mechanical systems on 4-Walker manifold. Also, the geometrical-physical results related to complex mechanical systems are also discussed for conformal Euler-Lagrangian equations.

Keywords: Walker Manifolds, Holomorphic, Symplectic Geometry, Conformal Geometry, Lagrangian, Mechanical System, Riemannian Manifold, Almost Complex Manifolds.

Introduction

Differential geometry is a mathematical discipline such that using known the techniques of differential calculus, integral calculus, linear algebra and multilinear algebra to study problems in geometry. Also, a dynamical system is a concept in mathematics where a fixed rule describes how a point in a geometrical space depends on time. At any given time a dynamical system has a state given by a set of real numbers (a vector) that can be represented by a point in an appropriate state space or a geometrical manifold. in addition, a dynamical systems theory is an area of mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. We can say that differential geometry provides a good workplace for studying Lagrangians mechanics of classical mechanics and field theory. The dynamic equations for moving bodies are obtained for Lagrangian mechanics by many authors in many areas [1-3]. *Kasap* and *Tekkoyun* found Lagrangian and Hamiltonian formalism for mechanical systems using para/pseudo-Kähler manifolds, representing an interesting multidisciplinary field of research [4]. *Walker* investigated a semi-Riemannian manifold which admits a field of parallel null r-planes, with $r \leq n/2$ [5].

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Salimov, Iscan and *Akbulut* examined that a Walker 4-manifold is a pseudo-Riemannian manifold, (M_4, g) of neutral signature, which admits a field of parallel null 2-plane [6]. *Matsushita* studied when these two almost complex structures are integrable and when the corresponding Kähler forms are symplectic and he obtained a useful method of constructing indefinite Kähler 4-manifolds with a Walker 4-manifold [7]. *Garcia-Rio* et al showed that such a Walker 4-manifold can carry various structures with respect to a certain kind of almost complex structure, e.g., symplectic structures, Kähler structures, Hermitian structures, according as the properties of certain functions which define the canonical form of the metric [8]. *Nadjafikhah* and *Jafari* constructed the optimal system of one-dimensional Lie subalgebras and investigate some of its group invariant solutions and they determined general form of four-dimensional Einstein Walker manifold [9]. *Salimov* and *Iscan* showed that a Walker 4-manifold is a semi-Riemannian manifold (M_4, g) of neutral signature, which admits a field of parallel null 2-plane [10]. *Brozos-Vazquez* et al examined commutativity properties of the Ricci operator, of the skew-symmetric curvature operator, and of the Jacobi operator for certain Walker manifolds of signature (2,2) [11]. *Davidov* shown that any proper almost Hermitian structure on a Walker 4-manifold is isotropic Kähler [12]. *Tekkoyun* shown that a Walker n-manifold is a semi-Riemannian n-manifold, which admits a field of parallel null *r*-planes, with $r \leq n/2$ [13]. *Ghanam* and *Thompson* gave an application of such a 4-dimensional Walker metric [14].

Preliminaries

Definition 1. Walker manifold is a triple (M, g, D) where M is an n-dimensional manifold, g is an indefinite metric and D is an r-dimensional parallel null distribution.

Of special interest are those manifolds admitting a field of null planes of maximum dimensionality $r = n/2$. Since the dimension of a null plane is source $r \leq n/2$, the lowest possible case is that of $(+,+,-,-)$ -manifolds admitting a field of parallel null 2-planes.

Definition 2. A **metric tensor** is a non-degenerate, smooth, symmetric, bilinear map which assigns a real number to pairs of tangent vectors at each tangent space of the manifold. Denoting the metric tensor g we can express this as $g: T_pM \times T_pM \to \mathbb{R}$. The map is symmetric and bilinear so if $X, Y, Z \in T_pM$ are tangent vectors at a point p to the manifold M then we have:

(1) $g(X,Y) = g(Y,X)$,

(2) $g(aX+Y,Z) = ag(X,Z) + g(Y,Z)$ for any real number $a \in \mathbb{R}$.

Definition 3. A **pseudo-Riemannian manifold** (also called a semi-Riemannian manifold) (M, g) is a differentiable manifold M equipped with a non-degenerate, smooth, symmetric metric tensor g.

Such a metric is called a pseudo-Riemannian metric and its values can be positive, negative or zero. The signature of a pseudo-Riemannian metric is (p,q), where both p and q are non-negative. The model space for a pseudo-Riemannian manifold of signature (p,q) is $\mathbb{R}^{p,q}$ with the metric $g = dx_1^2 + dx_2^2 + \ldots + dx_p^2 - dx_{p+1}^2 - \ldots - dx_{p+q}^2$.

Definition 4. Let M be a pseudo-Riemannian manifold of signature (p,q). We suppose given a splitting of the tangent bundle in the form $TM = V_1 \oplus V_2$ where V_1 and V_2 are smooth subbundles which are called distributions. If M is Riemannian, we can take $V_2 = V_1^{\perp}$ to be the orthogonal complement of V_1 and in that case V_2 is again parallel. In the pseudo-Riemannian setting, of course, $V_2 \cap V_1$ need not be trivial and there exist examples where although V_1 is parallel, there exists no complementary parallel distribution. Let V_1 be a parallel distribution. The rank of g restricted to V_1 is constant. We can say that V_1 is **a null parallel distribution** if V_1 is parallel and if the metric restricted to V_1 vanishes identically.

Proposition 1. A neutral metric g on a 4-manifold M_4 is said to be Walker metric if there exists a 2-dimensional null distribution D on M_4, which is parallel with respect to g. From Walker theorem there is a system of coordinates with respect to which g takes the local canonical form

$$g(x_1, x_2, x_3, x_4) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}, \tag{1}$$

where a, b, c are smooth functions of the coordinates (x_1, x_2, x_3, x_4). Also, $h_{x_1,\ldots,x_r}$ means partial derivatives $\frac{\partial h}{\partial x_1 \ldots \partial x_x}$ for any function $h(x_1, x_2, x_3, x_4)$. The parallel null 2-plane D is spanned locally by $\{\partial_{x_1}, \partial_{x_2}\}$, where ∂_{x_i} are abbreviated forms of $\partial_{x_1} = \frac{\partial}{\partial x_1}, \ldots, \partial_{x_4} = \frac{\partial}{\partial x_4}$ [5, 13]. Walker [5] studied pseudo-Riemannian manifolds M with a parallel field of null planes D and derived a canonical form. Motivated by this seminal work, one says that a pseudo-Riemannian manifold M which admits a null parallel i.e., (degenerate) distribution D is a Walker manifold.

Also, $\mathcal{F}(M_4)$, $\chi(M_4)$ and $\Lambda^1(M_4)$ are the set of functions on M_4, the set of vector fields on M_4 and the set of 1-forms on M_4, respectively. M_4 is a Walker manifold.

Conformal Geometry

The approach for studying conformal field theories is somewhat different from the usual approach for quantum and electromagnetic field theories.

Definition 5. A **conformal map or transformations** is a function which preserves angles.

It is well-known that in the most common case the function is between domains in the complex plane. Conformal maps can be defined between domains in higher dimensional Euclidean spaces, and more generally on a Riemann or semi-Riemann manifold. Conformal geometry is the study of the set of angle-preserving (conformal) transformations on a space. In two real dimensions, conformal geometry is precisely the geometry of Riemann surfaces.

Theorem 1. A **conformal manifold** is a differentiable manifold equipped with an equivalence class of Riemann metric tensors, in which two metrics g_1 and g_2 are equivalent if and only if

$$g_2 = \Psi^2 g_1, \tag{2}$$

where $\Psi > 0$ is a smooth positive function. An equivalence class of such metrics is known as a conformal metric or conformal class [15].

Theorem 2. A **conformal transformation** is a change of coordinates $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$ such that the metric changes by

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma). \tag{3}$$

A conformal field theory (CFT) is a field theory which is invariant under these transformations. Conformal field theories cares about angles, but not about distances. Transformation of the form (3) has a different interpretation depending on whether we are considering a fixed background metric $g_{\alpha\beta}$, or a dynamical background metric. When the metric is dynamical, the transformation is a diffeomorphism; this is a gauge symmetry. When the background is fixed, physical symmetry, taking the point σ^α to point $\tilde{\sigma}^\alpha$. This is now a global symmetry with the corresponding conserved currents.

The Theory of *J*-Holomorphic Curves

Definition 6. ***J*–holomorphic curve** is a smooth map from a Riemann surface into an almost complex manifold that satisfies the Cauchy–Riemann equation.

The theory of *J*–holomorphic curves is one of the new techniques which have recently revolutionized the study of symplectic geometry, making it possible to study the global structure of symplectic manifolds. The methods are also of interest in the study of Kähler manifolds, since often when one studies properties of holomorphic curves in such manifolds it is necessary to perturb the complex structure to be generic. The effect of this is to ensure that one is looking at persistent rather than accidental features of these curves.

Definition 7. A **symplectic manifold** is a smooth manifold (M) equipped with a closed nondegenerate differential 2-form (ω) called the symplectic form.

The study of symplectic manifolds is called symplectic geometry. Symplectic manifolds arise naturally in abstract formulations of classical mechanics and analytical mechanics as the cotangent bundles of manifolds, e.g., in the Hamiltonian formulation of classical mechanics, which provides one of the major motivations for the field: The set of all possible configurations of a system is modelled as a manifold, and this manifold's cotangent bundle describes the phase space of the system.

Example 1. An almost complex symplectic manifold is standard Euclidean space $(\mathbb{R}^{2n},\omega_0)$ with its standard almost complex structure J_0 obtained from the usual identification with $\mathbb{C}_n$. Thus, one sets $z_j = x_{2j-1} + ix_{2j}$ for $j = 1,...,n$ and defines J_0 by

$$J_0\left(\partial_{2j-1}\right) = \partial_{2j}\ ,\quad J_0\left(\partial_{2j}\right) = -\partial_{2j-1}\ , \tag{4}$$

where $\partial_j = \partial / \partial x_j$ is the standard basis of $T_x\mathbb{R}^{2n}$ [16].

Almost (para)-Complex, Tangent Structures and Manifolds

Definition 8. Let M be a smooth manifold of real dimension 2n. We say that a smooth atlas A of M is holomorphic if for any two coordinate charts $z : U \to U' \subset \mathbb{C}^m$ and $w : V \to V' \subset \mathbb{C}^m$ in A, the coordinate transition map $z \circ w^{-1}$ is holomorphic. Any holomorphic atlas uniquely determines a maximal holomorphic atlas, and a maximal holomorphic atlas is called a **complex structure** for M. We say that M is a **complex manifold** of complex dimension n if M comes equipped with a holomorphic atlas. Any coordinate chart of the corresponding complex structure will be called a holomorphic coordinate chart of M. A Riemann surface or complex curve is a complex manifold of complex dimension 1.

Definition 9. Let M be a differentiable manifold of dimension 2n, and suppose J is a differentiable vector bundle isomorphism $J : TM \to TM$ such that $J_x : T_xM \to T_xM$ is a **complex structure** for T_xM, i.e. $J^2 I = -I$ where I is the identity (unit) operator on V. Then J is called an almost-complex structure for the differentiable manifold M. A manifold with a fixed **almost complex structure** is called an **almost complex manifold.**

A celebrated theorem of Newlander and Nirenberg [17] says that an almost (para) complex structure is a (para) complex structure if and only if its Nijenhuis tensor or torsion vanishes.

Theorem 3. The almost (para) complex structure J on M is integrable if and only if the tensor N_J vanishes identically, where N_J is defined on two vector fields X and Y by

$$N_J[X,Y]=[JX,JY]-J[X,JY]-J[JX,Y]-[X,Y]. \quad (5)$$

The tensor (2,1) is called the Nijenhuis tensor (5). We say that J is torsion free if $N_J=0$.

Properties of Almost Complex Structure *J*

Example 3. Let O be an open subset of $\mathbb{R}^4$. Let $a,b,c\in C^\infty(O)$ be smooth function on $O.$ We set $M_{a,b,c}:=(O,g_{a,b,c})$ where

$$\begin{aligned} g_{a,b,c} &:= 2(dx_1\circ dx_3+dx_2\circ dx_4)+a(x_1,x_2,x_3,x_4)dx_3\circ dx_3 \\ &+b(x_1,x_2,x_3,x_4)dx_{34}\circ dx_4+2c(x_1,x_2,x_3,x_4)dx_3\circ dx_4. \end{aligned} \quad (6)$$

Example 4. Let $M_{a,b,c}:=(O,g_{a,b,c})$ be the pseudo-Riemannian manifold of (6) and we take the induced orthonormal basis;

$$\begin{aligned} e_1 &= \frac{1}{2}(1-a)\partial_{x_1}+\partial_{x_3}, \quad & e_2 &= -c\partial_{x_2}+\frac{1}{2}(1-b)\partial_{x_4}, \\ e_3 &= -\frac{1}{2}(1+a)\partial_{x_1}+\partial_{x_3}, \quad & e_4 &= -c\partial_{x_2}-\frac{1}{2}(1+b)\partial_{x_4}. \end{aligned} \quad (7)$$

We shall let $C_{a,b,c}:=(O,g_{a,b,c},J)$ where J is the proper almost complex structure defined by:

$$J=e_2\otimes e^1-e_1\otimes e^2+e_4\otimes e^3-e_3\otimes e^4. \quad (8)$$

Thus, the following expression is obtained:

$$J:e_1=e_2, J:e_2=-e_1, J:e_3=e_4, J:e_4=-e_3. \quad (9)$$

The map J induces a positive $\frac{\pi}{2}$-rotation on the degenerate parallel field $D:=Span\{\partial x_1,\partial x_2\}$:

$$\begin{aligned} J\partial_{x_1} &= \partial_{x_2}, \quad & J\partial_{x_3} &= -c\partial_{x_1}+\frac{1}{2}(a-b)\partial_{x_2}+\partial_{x_4}, \\ J\partial_{x_2} &= -\partial_{x_1}, \quad & J\partial_{x_4} &= \frac{1}{2}(a-b)\partial_{x_1}+c\partial_{x_2}-\partial_{x_3}. \end{aligned} \quad (10)$$

The above structures were taken from [12, 18]. The following result shows that the class of isotropic Kähler structures is larger than might at first sight be expected:

Theorem 4. Let $C_{a,b,c} := O, g_{a,b,c}$ be as given in **Example 4.**

(1) $C_{a,b,c}$ is almost Kähler if and only if $a_1 + b_1 = 0$ and $a_2 + b_2 = 0$.

(2) $C_{a,b,c}$ is Hermitian if and only if $a_1 + b_1 = 2c_2$ and $a_2 - b_2 = -2c_1$.

(3) $C_{a,b,c}$ is is Kähler if and only if $a_1 = -b_1 = c_2$ and $a_2 = -b_2 = -c_1$.

Definition 10. In three dimensions, **the vector from the origin to the point** with cartesian coordinates (x, y, z) can be written as [19]: $r = x\vec{i} + y\vec{j} + z\vec{k} = x\left(\frac{\partial}{\partial x}\right) + y\left(\frac{\partial}{\partial y}\right) + z\left(\frac{\partial}{\partial z}\right)$.

Proposition 1. Conformal to the structure coefficient $\Psi = \Psi(x_1, x_2, x_3, x_4)$, J holomorphic property preservation and using **Theorems 1** and **2**, is proposed as follows.

$$
\begin{aligned}
&J\frac{\partial}{\partial_{x_1}} = \Psi^2 \frac{\partial}{\partial_{x_2}}, \quad J\frac{\partial}{\partial_{x_3}} = -c\Psi^{-2}\frac{\partial}{\partial_{x_1}} + \frac{1}{2}(a-b)\Psi^2\frac{\partial}{\partial_{x_2}} + \Psi^2\frac{\partial}{\partial_{x_4}},\\
&J\frac{\partial}{\partial_{x_2}} = -\Psi^{-2}\frac{\partial}{\partial_{x_1}}, \quad J\frac{\partial}{\partial_{x_4}} = \frac{1}{2}(a-b)\Psi^{-2}\frac{\partial}{\partial_{x_1}} + c\Psi^{-2}\frac{\partial}{\partial_{x_2}} - \Psi^{-2}\frac{\partial}{\partial_{x_3}}.
\end{aligned}
\tag{11}
$$

Proof :

$$(1)\ J^2\frac{\partial}{\partial_{x_1}} = \Psi^2 J\left(\frac{\partial}{\partial_{x_2}}\right) = -\frac{\partial}{\partial_{x_1}},$$

$$(2)\ J^2\frac{\partial}{\partial_{x_2}} = -\Psi^{-2} J\left(\frac{\partial}{\partial_{x_1}}\right) = -\frac{\partial}{\partial_{x_2}},$$

$$
\begin{aligned}
(3)\ J^2\frac{\partial}{\partial_{x_3}} &= -c\Psi^{-2}J\left(\frac{\partial}{\partial_{x_1}}\right) + \frac{1}{2}(a-b)\Psi^2 J\left(\frac{\partial}{\partial_{x_2}}\right) + \Psi^2 J\left(\frac{\partial}{\partial_{x_4}}\right)\\
&= -c\Psi^{-2}\Psi^2\frac{\partial}{\partial_{x_2}} - \frac{1}{2}(a-b)\Psi^{-2}\Psi^2\frac{\partial}{\partial_{x_1}}\\
&+\Psi^2\left[\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial}{\partial_{x_1}} + c\Psi^{-2}\frac{\partial}{\partial_{x_2}} - \Psi^{-2}\frac{\partial}{\partial_{x_3}}\right]\\
&= -c\frac{\partial}{\partial_{x_2}} - \frac{1}{2}(a-b)\frac{\partial}{\partial_{x_1}} + \frac{1}{2}(a-b)\frac{\partial}{\partial_{x_1}} + c\frac{\partial}{\partial_{x_2}} - \frac{\partial}{\partial_{x_3}} = -\frac{\partial}{\partial_{x_3}},
\end{aligned}
\tag{12}
$$

$$(4)\ J^2\frac{\partial}{\partial_{x_4}}=\frac{1}{2}\left(a-b\right)\Psi^{-2}J(\frac{\partial}{\partial_{x_1}})+c\Psi^{-2}J\left(\frac{\partial}{\partial_{x_2}}\right)-\Psi^{-2}J\left(\frac{\partial}{\partial_{x_3}}\right)$$

$$=\frac{1}{2}\left(a-b\right)\Psi^{-2}\Psi^{2}\frac{\partial}{\partial_{x_2}}-c\frac{1}{\Psi^4}\frac{\partial}{\partial_{x_1}}$$

$$-\Psi^{-2}\left[-c\Psi^{-2}\frac{\partial}{\partial_{x_1}}+\frac{1}{2}\left(a-b\right)\Psi^{2}\frac{\partial}{\partial_{x_2}}+\Psi^{2}\frac{\partial}{\partial_{x_4}}\right]$$

$$=\frac{1}{2}\left(a-b\right)\Psi^{-2}\Psi^{2}\frac{\partial}{\partial_{x_2}}-c\Psi^{-4}\frac{\partial}{\partial_{x_1}}+c\Psi^{-4}\frac{\partial}{\partial_{x_1}}$$

$$-\frac{1}{2}\left(a-b\right)\Psi^{-2}\Psi^{2}\frac{\partial}{\partial_{x_2}}-\Psi^{-2}\Psi^{2}\frac{\partial}{\partial_{x_4}}=-\frac{\partial}{\partial_{x_4}}.$$

As seen above, holomorphic structures ($J^2\frac{\partial}{\partial_{x_i}}=-\frac{\partial}{\partial_{x_i}}$ or $J^2 I=-I$) are complex.

Lagrange Dynamics Equations

Theorem 5. The closed 2-form (ω) on a vector field (ξ) and $1-$form reduction function (i_ξ) on the phase space defined of a mechanical system ($i_\xi\omega$) is equal to the differential of the energy function 1-form (dE) of the Lagrangian and the Hamiltonian mechanical systems [1, 20].

Definition 11. Let M be an *n*-dimensional manifold and TM its tangent bundle with canonical projection $\tau_M : TM \to M$. TM is called the phase space of velocities of the base manifold M. Let $L : TM \to R$ be a differentiable function on TM called the **Lagrangian function**. Here, $L = T - V$ such that T is the kinetic energy and V is the potential energy of a mechanical system. In the problem of a mass on the end of a spring, $T = m\dot{x}^2/2$ and $V = kx^2/2$, so we have $L=\frac{m\dot{x}^2}{2}-\frac{kx^2}{2}$. We consider the closed 2-form and base space (J) on TM given by $\Phi_L=-d\mathbf{d}_J L=-d\left(J(\mathbf{d})\right)$. Consider the equation

$$i_\xi\Phi_L=dE_L. \tag{13}$$

where i_ξ is reduction function and $i_\xi\Phi_L=\Phi_L(\xi)$ is defined in the form. Then ξ is a vector field, we shall see that (13) under a certain condition on ξ is the intrinsical expression of the Euler-Lagrange equations of motion. This equation (13) is named as **Lagrange Dynamical Equation**.

Definition 12. We shall see that for motion in a potential, $E_L = VL - L$ is an energy function and $V = J\xi$ a Liouville vector field. Here dE_L denotes the differential of E. The triple (TM, Φ_L, ξ) is known as **Lagrangian system** on the tangent bundle TM. If it is continued the operations on (13) for any coordinate system then infinite dimension **Lagrange's equation is** obtained the form below. The equations of motion in Lagrangian mechanics are the Lagrange equations of the second kind, also known as the Euler–Lagrange equations;

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}. \tag{14}$$

Definition 13. We have $\frac{\partial L}{\partial \dot{x}} = mx$ and $\frac{\partial L}{\partial x} = -kx$, so eq. (14) gives $m\ddot{x} = -kx$ which is exactly the result obtained by using $F = ma$ at Newton's second law for the mechanical problem. The Euler-Lagrange equation, eq. (14), gives $m\ddot{x} = -\frac{dV}{dx}$. In a three-dimensional setup written in terms of cartesian coordinates, the potential takes the form $V(x,y,z)$, so the Lagrangian is $L = \frac{m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)}{2} - V(x,y,z)$. So, the three Euler-Lagrange equations may be combined into the vector statement $m\ddot{x} = -\nabla V$.

Conformal Euler-Lagrange Equations

Proposition 2. Let M_4 be a Walker manifold and $\{x_1, x_2, x_3, x_4\}$ be its coordinate functions. Let ξ be a the semispray vector field determined by

$$\xi = \sum_{i=1}^{4} X^i \partial_{x_i}, \tag{15}$$

where $\sum_{i=1}^{4} X^i = \dot{x}_i$ and the dot indicates the derivative with respect to time t.

Euler Lagrange equations are obtained by using **Theorem 2**, **Theorem 5** and (13). By means of the proper almost complex structure J given by (11), the vector field is defined by

$$\begin{aligned} V &= J(\xi) \\ &= X^1\Psi^2 \frac{\partial}{\partial_{x_2}} - X^2\Psi^{-2}\frac{\partial}{\partial_{x_1}} + X^3\left(-c\Psi^{-2}\frac{\partial}{\partial_{x_1}} + \frac{1}{2}(a-b)\Psi^2\frac{\partial}{\partial_{x_2}} + \Psi^2\frac{\partial}{\partial_{x_4}}\right) \\ &\quad + X^4\left(\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial}{\partial_{x_1}} + c\Psi^{-2}\frac{\partial}{\partial_{x_2}} - \Psi^{-2}\frac{\partial}{\partial_{x_3}}\right), \end{aligned} \tag{16}$$

which is named Liouville vector field on the Walker manifold M_4. The maps given by $T, P : M_4 \to \mathbb{R}$

such that $T=\frac{1}{2}m_i(\dot{x}_1^{\,2}+\dot{x}_2^2+\dot{x}_3^2+\dot{x}_4^2)$, $V=m_i gh$ are said to be the kinetic energy and the potential energy of the system, respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on Walker manifold M_4, respectively. Then $L:M_4\to\mathbb{R}$ is a map that satisfies the conditions; i) $L=T-V$ is a Lagrangian function, ii) the function determined by $E_L=VL-L,$ is energy function. The function i_J induced by J and denoted by

$$i_J\omega(X_1,X_2,...,X_r)=\sum_{i=1}^{r}J(X_1,...,JX_i,...,X_r), \tag{17}$$

is called vertical derivation, where $\omega\in\wedge^r M_4,$ $X_i\in\chi(M_4).$ The vertical differentiation d_J is given by $\mathbf{d}_J=[i_J,d]=i_Jd-di_J,$ where d is the usual exterior derivation. For the almost complex structure J given by (11), the form on Walker manifold M_4 is the closed 2-form determined by $\Phi_L=-d\mathbf{d}_JL$ such that $d_J:\mathcal{F}(M_4)\to\wedge^1 M_4,$

$$\mathbf{d}=\sum_{i=1}^{4}\frac{\partial}{\partial_{x_i}}dx_i, \text{ and}$$

$$\begin{aligned}\mathbf{d}_J=J(\mathbf{d})&=\Psi^2\frac{\partial}{\partial_{x_2}}dx_1-\Psi^{-2}\frac{\partial}{\partial_{x_1}}dx_2\\&+\left(-c\Psi^{-2}\frac{\partial}{\partial_{x_1}}+\frac{1}{2}(a-b)\Psi^2\frac{\partial}{\partial_{x_2}}+\Psi^2\frac{\partial}{\partial_{x_4}}\right)dx_3\\&+\left(\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial}{\partial_{x_1}}+c\Psi^{-2}\frac{\partial}{\partial_{x_2}}-\Psi^{-2}\frac{\partial}{\partial_{x_3}}\right)dx_4.\end{aligned} \tag{18}$$

Now, we will calculate the first part of (13). Through a direct computation using (18), the closed 2-form Φ_L is seen to be as follows:

$$\Phi_L=\sum_{i=1}^{4}\left[\begin{array}{c}\left(\Psi^2\frac{\partial^2 L}{\partial x_i\partial x_2}+2\Psi\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_2}\right)dx_1\wedge dx_i\\ +\left(-\Psi^{-2}\frac{\partial^2 L}{\partial x_i\partial x_1}+2\Psi^{-3}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_1}\right)dx_2\wedge dx_i\\ +\left(\begin{array}{c}-c\Psi^{-2}\frac{\partial^2 L}{\partial x_i\partial x_1}+2c\Psi^{-3}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_1}+\frac{1}{2}(a-b)\Psi^2\frac{\partial^2 L}{\partial x_i\partial x_2}\\ +2\frac{1}{2}(a-b)\Psi\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_2}+\Psi^2\frac{\partial^2 L}{\partial x_i\partial x_4}+2\Psi\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_4}\end{array}\right)dx_3\wedge dx_i\\ +\left(\begin{array}{c}\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial^2 L}{\partial x_i\partial x_1}-2\frac{1}{2}(a-b)\Psi^{-3}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_1}+c\Psi^{-2}\frac{\partial^2 L}{\partial x_i\partial x_2}\\ -2c\Psi^{-3}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_2}-\Psi^{-2}\frac{\partial^2 L}{\partial x_i\partial x_3}+2\Psi^{-3}\frac{\partial\Psi}{\partial x_i}\frac{\partial L}{\partial x_3}\end{array}\right)dx_4\wedge dx_i\end{array}\right], \tag{19}$$

and $\Phi_L(\xi)$;

$$-X^1\left(\Psi^2\frac{\partial^2 L}{\partial x_1\partial x_2}+2\Psi\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_2}\right)dx_1-X^1\left(-\Psi^{-2}\frac{\partial^2 L}{\partial x_1\partial x_1}+2\Psi^{-3}\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_1}\right)dx_2$$

$$-X^1\left(\begin{array}{c}-c\Psi^{-2}\frac{\partial^2 L}{\partial x_1\partial x_1}+2c\Psi^{-3}\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_1}+\frac{1}{2}(a-b)\Psi^2\frac{\partial^2 L}{\partial x_1\partial x_2}\\ +2\frac{1}{2}(a-b)\Psi\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_2}+\Psi^2\frac{\partial^2 L}{\partial x_1\partial x_4}+2\Psi\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_4}\end{array}\right)dx_3$$

$$-X^1\left(\begin{array}{c}\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial^2 L}{\partial x_1\partial x_1}-2\frac{1}{2}(a-b)\Psi^{-3}\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_1}+c\Psi^{-2}\frac{\partial^2 L}{\partial x_1\partial x_2}\\ -2c\Psi^{-3}\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_2}-\Psi^{-2}\frac{\partial^2 L}{\partial x_1\partial x_3}+2\Psi^{-3}\frac{\partial\Psi}{\partial x_1}\frac{\partial L}{\partial x_3}\end{array}\right)dx_4$$

$$-X^2\left(\begin{array}{c}-c\Psi^{-2}\frac{\partial^2 L}{\partial x_2\partial x_1}+2\Psi^{-3}\frac{\partial\Psi}{\partial x_2}\frac{\partial L}{\partial x_1}+\frac{1}{2}(a-b)\Psi^2\frac{\partial^2 L}{\partial x_2\partial x_2}\\ +2\frac{1}{2}(a-b)\Psi\frac{\partial\Psi}{\partial x_2}\frac{\partial L}{\partial x_2}+\Psi^2\frac{\partial^2 L}{\partial x_2\partial x_4}+2\Psi\frac{\partial\Psi}{\partial x_2}\frac{\partial L}{\partial x_4}\end{array}\right)dx_3$$

$$-X^2\begin{pmatrix}\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial^2 L}{\partial_{x_2}\partial_{x_1}}-2\frac{1}{2}(a-b)\Psi^{-3}\frac{\partial\Psi}{\partial_{x_2}}\frac{\partial L}{\partial_{x_1}}\\ +c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_2}\partial_{x_2}}-2c\Psi^{-3}\frac{\partial\Psi}{\partial_{x_2}}\frac{\partial L}{\partial_{x_2}}-\Psi^{-2}\frac{\partial^2 L}{\partial_{x_2}\partial_{x_3}}+2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_2}}\frac{\partial L}{\partial_{x_3}}\end{pmatrix}dx_4 \tag{20}$$

$$-X^3\left(\Psi^2\frac{\partial^2 L}{\partial_{x_3}\partial_{x_2}}+2\Psi\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_2}}\right)dx_1-X^3\left(-\Psi^{-2}\frac{\partial^2 L}{\partial_{x_3}\partial_{x_1}}+2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_1}}\right)dx_2$$

$$-X^3\begin{pmatrix}-c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_3}\partial_{x_1}}+2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_1}}+\frac{1}{2}(a-b)\Psi^2\frac{\partial^2 L}{\partial_{x_3}\partial_{x_2}}\\ +2\frac{1}{2}(a-b)\Psi\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_2}}+\Psi^2\frac{\partial^2 L}{\partial_{x_3}\partial_{x_4}}+2\Psi\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_4}}\end{pmatrix}dx_3$$

$$-X^3\begin{pmatrix}\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial^2 L}{\partial_{x_3}\partial_{x_1}}-2\frac{1}{2}(a-b)\Psi^{-3}\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_1}}+c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_3}\partial_{x_2}}\\ -2c\Psi^{-3}\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_2}}-\Psi^{-2}\frac{\partial^2 L}{\partial_{x_3}\partial_{x_3}}+2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_3}}\frac{\partial L}{\partial_{x_3}}\end{pmatrix}dx_4$$

$$-X^4\left(\Psi^2\frac{\partial^2 L}{\partial_{x_4}\partial_{x_2}}+2\Psi\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_2}}\right)dx_1-X^4\left(-\Psi^{-2}\frac{\partial^2 L}{\partial_{x_4}\partial_{x_1}}+2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_1}}\right)dx_2$$

$$-X^4\begin{pmatrix}-c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_4}\partial_{x_1}}+2c\Psi^{-3}\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_1}}+\frac{1}{2}(a-b)\Psi^2\frac{\partial^2 L}{\partial_{x_4}\partial_{x_2}}\\ +2\frac{1}{2}(a-b)\Psi\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_2}}+\Psi^2\frac{\partial^2 L}{\partial_{x_4}\partial_{x_4}}+2\Psi\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_4}}\end{pmatrix}dx_3$$

$$-X^4\begin{pmatrix}\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial^2 L}{\partial_{x_4}\partial_{x_1}}-2\frac{1}{2}(a-b)\Psi^{-3}\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_1}}+c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_4}\partial_{x_2}}\\ -2c\Psi^{-3}\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_2}}-\Psi^{-2}\frac{\partial^2 L}{\partial_{x_4}\partial_{x_3}}+2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_4}}\frac{\partial L}{\partial_{x_3}}\end{pmatrix}dx_4. \tag{21}$$

Then the energy function E_L is found as follows:

$$\begin{aligned} E_L = V(L) - L = X^1\Psi^2\frac{\partial L}{\partial_{x_2}} - X^2\Psi^{-2}\frac{\partial L}{\partial_{x_1}} \\ +X^3\left(-c\Psi^{-2}\frac{\partial L}{\partial_{x_1}} + \frac{1}{2}(a-b)\Psi^2\frac{\partial L}{\partial_{x_2}} + \Psi^2\frac{\partial L}{\partial_{x_4}}\right) \\ +X^4\left(\frac{1}{2}(a-b)\Psi^{-2}\frac{\partial L}{\partial_{x_1}} + c\Psi^{-2}\frac{\partial L}{\partial_{x_2}} - \Psi^{-2}\frac{\partial L}{\partial_{x_3}}\right) - L. \end{aligned} \tag{22}$$

Now, we will calculate the second part of (13). Thus, the differential energy function is as follows:

$$dE_L = \sum_{i=1}^{4}\left[\begin{aligned} & X^1\left[\Psi^2\frac{\partial^2 L}{\partial_{x_i}\partial_{x_2}}dx_i + 2\Psi\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_2}}dx_i\right] \\ & -X^2\left[\Psi^{-2}\frac{\partial^2 L}{\partial_{x_i}\partial_{x_i}}dx_i - 2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_i}}dx_i\right] \\ & +X^3\left[\begin{aligned} & -c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_i}\partial_{x_i}}dx_i + 2c\Psi^{-3}\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_i}}dx_i + \frac{1}{2}(a-b)\Psi^2\frac{\partial^2 L}{\partial_{x_i}\partial_{x_2}}dx_i \\ & +\frac{1}{2}(a-b)2\Psi\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_2}}dx_i + \Psi^2\frac{\partial^2 L}{\partial_{x_i}\partial_{x_4}}dx_i + 2\Psi\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_4}}dx_i \end{aligned}\right] \\ & +X^4\left[\begin{aligned} & \frac{1}{2}(a-b)\Psi^{-2}\frac{\partial^2 L}{\partial_{x_i}\partial_{x_1}}dx_i - \frac{1}{2}(a-b)2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_i}}dx_i \\ & +c\Psi^{-2}\frac{\partial^2 L}{\partial_{x_i}\partial_{x_2}}dx_i - c2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_2}}dx_i \\ & -\Psi^{-2}\frac{\partial^2 L}{\partial_{x_i}\partial_{x_3}}dx_i + 2\Psi^{-3}\frac{\partial\Psi}{\partial_{x_i}}\frac{\partial L}{\partial_{x_3}}dx_i \end{aligned}\right] - \frac{\partial L}{\partial_{x_i}}dx_i \end{aligned}\right]. \tag{23}$$

Suppose that a curve $\alpha : \mathbb{R} \to M_4$ be an integral curve of semispray ξ. According to (2), using (18) and (23) then we find the following equations:

$$
\begin{aligned}
&dif1: && -\frac{\partial}{\partial t}\left(\Psi^{2}\frac{\partial L}{\partial_{x_2}}\right)+\frac{\partial L}{\partial_{x_1}}=0,\\
&dif2: && \frac{\partial}{\partial t}\left(\Psi^{-2}\frac{\partial L}{\partial_{x_1}}\right)+\frac{\partial L}{\partial_{x_2}}=0,\\
&dif3: && c\frac{\partial}{\partial t}\left(\Psi^{-2}\frac{\partial L}{\partial_{x_1}}\right)-\frac{1}{2}(a-b)\frac{\partial}{\partial t}\left(\Psi^{2}\frac{\partial L}{\partial_{x_2}}\right)-\frac{\partial}{\partial t}\left(\Psi^{2}\frac{\partial L}{\partial_{x_4}}\right)+\frac{\partial L}{\partial_{x_3}}=0,\\
&dif4: && -\frac{1}{2}(a-b)\frac{\partial}{\partial t}\left(\Psi^{-2}\frac{\partial L}{\partial_{x_1}}\right)-c\frac{\partial}{\partial t}\left(\Psi^{-2}\frac{\partial L}{\partial_{x_2}}\right)+\frac{\partial}{\partial t}\left(\Psi^{-2}\frac{\partial L}{\partial_{x_3}}\right)+\frac{\partial L}{\partial_{x_4}}=0,
\end{aligned}
\tag{24}
$$

such that the equations calculated in (24) are named *complex conformal Euler-Lagrange equations* constructed on Walker manifold M_4 if 2-form Φ_L is symplectic structure. Thus the triple (M_4, Φ_L, ξ) is named a **complex conformal Euler-Lagrange mechanical system** on Walker manifold M_4.

Computer Solution of Equations and Graph

It is well-known that an electromagnetic field is a physical field produced by electrically charged objects. How the movement of objects in electrical, magnetically and gravitational fields force is very important. For instance, on a weather map and the surface wind velocity is defined by assigning a vector to each point on a map. Also, each vector represents the speed and direction of the movement of air at that point.

These found (24) are partial differential equation. We can solve these equations systems using software for

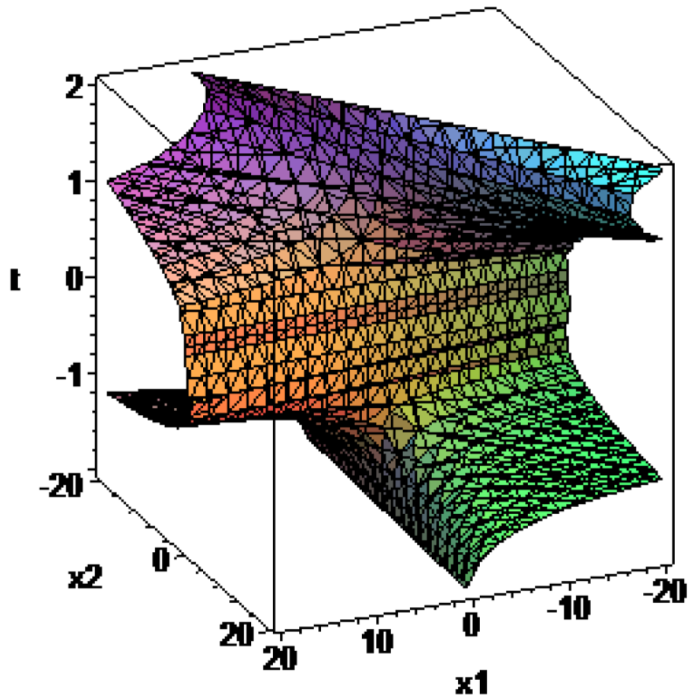
Theorem 4. (1) $C_{a,b,c}$ almost Kähler. It implicit solution is as follows:

$$
\begin{aligned}
L(x_1,x_2,x_3,x_4,t) := (((x_1*c_2+c_3*x_3)*t^3+(-c_4*x_4-x_3*c_1-x_2*c_1)\\
*t+x_2*c_2+x_4*c_3+x_3*c_2)*cos(t)\\
+((x_1*c_1+c_4*x_3)*t^3+(x_2*c_2+x_4*c_3+x_3*c_2)*t\\
+c_4*x_4+x_3*c_1+x_2*c_1)*sin(t)+F_5(t)*t^2)/t^2.
\end{aligned}
\tag{25}
$$

The location of each object in space represented by three dimensions in physical space. Three-dimensional space is a geometric three-parameter model of the physical universe in which all known matter exists. These three dimensions can be labeled by a combination of three chosen from the terms length, width, height, depth, and breadth. Any three directions can be chosen, provided that they do not all lie in the same plane. So, each vector represents the speed and direction of the movement of air at that point. The number of dimensions of the equation (25) will be reduced to three and behind the graphics will be drawn. First, closed function at (25) will be selected as a special. After, the figure of the equation (25) has been drawn for the route of the movement of objects in the electromagnetic field.

Example 5. We choose at (25) as special case of $\Psi(x_1, x_2, x_3, x_4, t) = t$. Lagrange function and the graph is as follows:

$$L(x_1, x_2, x_3, x_4, t) := (((x_1+1)*t^3 + (-2-x_2)*t + x_2 + 2)*cos(t) \\ +((x_1+1)*t^3 + (x_2+2)*t + 2 + x_2)*sin(t) + t^3)/t^2, \quad (26)$$



Figure

Discussion

A classical field theory explain the study of how one or more physical fields interact with matter which is used quantum and classical mechanics of physics branches. A classical field theory explain the study of how one or more physical fields interact with matter which is used quantum and classical mechanics of physics branches. The most important advantage of this study is to obtain geodesic on 4-Walker manifolds. Thus, geodesics is to allow the calculation of linear or nonlinear distance for the orbits of moving objects. In addition, in the equations implicit solutions (25) were found using symbolic computation program. In this study, conformal Euler-Lagrange mechanical equations (24) derived on a generalized on 4-Walker manifolds may be suggested to deal with problems in electrical, magnetically and gravitational fields force for the path of movement (26) of defined space moving objects [2, 21].

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