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# Measuring Gap Risk for Constant Proportion Portfolio Insurance Strategies in Uncertain Markets

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## Abstract

Portfolio insurance is a critical component of portfolio management. Constant proportion portfolio insurance (CPPI) is one of the most popular and widely used portfolio insurance approaches. Recent years have witnessed increased application of CPPI. The notion of CPPI limits the downside risk of a portfolio when markets are bearish whilst maintaining its upside potential when markets are bullish. The practice behind CPPI is to shift financial resources between risky and risk-free asset classes. Conceptually, the cushion plays a central role in the dynamics of CPPI. Portfolio insurance is currently based on stochastic finance theory. Probability theory recognises randomness as the only important form of indeterminacy in asset pricing. However, recent research through uncertainty theory led to the birth of uncertain finance theory. Uncertainty theory proposes that uncertainty is the only legitimate form of indeterminacy which should be taken into account in asset pricing. Therefore, it is in the best interest of this research paper to analyse portfolio insurance in uncertain markets. In continuous time diffusion models, CPPI techniques are not exposed to gap risk. However, in practical reality CPPI approaches are exposed to gap risk. To model the dynamics of CPPI, the study adopts jump-diffusion models where the value of the underlying portfolio exhibits sudden significant downward shocks. The aim of this research paper is to quantify gap risk for CPPI strategies in uncertain markets using the investment risk index. The study also analyses the relationship between a pre-determined participation rate, $m$, and the value of the portfolio. The importance of $m$ in CPPI is also examined.

*Keywords*: Portfolio insurance, portfolio management, probability theory, uncertainty theory, indeterminacy, randomness, uncertainty, gap risk, uncertain markets, participation rate.

## Introduction

Since their introduction stochastic processes have been very useful in the field of finance. Stochastic processes deal with random phenomena. A Brownian motion is one of the most famous and widely used

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stochastic processes. Robert Brown studied the concept of a Brownian motion in 1827 and Einstein refined it in 1905. The notion of a Brownian motion was further fine-tuned by Norbert Wiener in 1923. A by-product of Wiener's work is a mathematical theory known as a Wiener process. Therefore, a Brownian motion is known as a Wiener process or a Wiener-Einstein process (Taylor and Karlin, 1998). Based on a Wiener process, stochastic calculus was pioneered by Ito in 1944. Stochastic calculus is a branch of pure mathematics that deals with the differentiation and integration of stochastic processes.

Bachelier, through some research work in the Paris stock market, was the first person to introduce a Brownian motion in finance in 1900. Bachelier assumed that stock prices follow a Brownian motion. This assumption proved to be unrealistic as a Brownian motion can predict non-positive stock prices. Against a background of Bachelier's work, Samuelson (1965) proposes that stock prices follow a geometric Brownian motion. However, Samuelson (1965) fails to take into account the time value of money, among other things. Black and Scholes (1973) also assume that stock prices follow a geometric Brownian motion and propose a famous and widely used Black-Scholes stock model. The Black-Scholes stock model is widely used in asset pricing and portfolio management. Merton (1973) supports the concept that stock prices follow a geometric Brownian motion as well.

Portfolio insurance refers to investment strategies which promise that the portfolio value at any time upto maturity will not fall below the lower bound (Cont and Tankov, 2007). Currently, portfolio insurance is based on stochastic finance theory. Probability theory occupies a central role in stochastic finance theory. The study of probability was pioneered by Pascal and Fermat (1654) and the foundation of probability theory was suggested by Kolmogorov (1933). Financial decisions are made under the conditions of indeterminacy. Peng (2013) defines indeterminacy as any phenomenon whose outcomes cannot precisely be determined in advance. In other words, indeterminacy can be defined as a condition or state of events' outcomes being unpredictable in advance.

It is documented that probability theory recognises randomness as the only form of indeterminacy which is critical in asset pricing. Randomness refers to any phenomenon which can be quantified by a probability measure (Liu, 2012). In other terms, randomness is a characteristic of anything which can be described by probability. Probability theory is used when the sample size is large enough (Liu, 2012). This indicates that probability theory is employed when the frequency is known and when the probabilitydistribution is available. Frequency is a factual element, it does not change with the change in knowledge and preferences. Probability theory is an axiomatic branch of pure mathematics which deals with random phenomena.

However, recent research has resulted in the emergence of uncertain finance theory which is founded on uncertainty theory. Uncertain finance theory is gaining more popularity with each passing day. The theory of uncertainty was first studied by Liu (2007) and further improved by Liu (2010). Uncertainty theory recognises uncertainty as the only legitimate form of indeterminacywhich is important as far as the pricing of assets is concerned (Liu, 2012). Any phenomenon which can be quantified by an uncertain measure is called uncertainty

(Liu, 2012). Uncertainty is an attribute of anything which can be described by belief degrees.

Personal belief degrees play a central role in the modelling of uncertainty. A personal belief degree refers to the strength with which participants believe that a certain event will occur. An uncertain measure shows a personal belief degree of an uncertain event that may occur (Liu, 2012). Personal belief degrees change as knowledge and preferences change. Uncertainty influences the behaviour of market participants. Therefore, uncertainty creates frictions in financial markets. Conceptually, uncertainty theory is used when the sample size is too small (or even no sample). Uncertainty theory is adopted when the frequency is not known and when the uncertainty distribution is not available.

Uncertainty theory is an axiomatic branch of pure mathematics for modelling human uncertainty. Liu (2008) proposes a notion of uncertain process which models the dynamics of uncertain phenomena that vary with time. An uncertain process is defined as a sequence of uncertain variables which vary with time (Liu, 2012). A canonical process and a renewal process are two important types of uncertain processes.

A renewal process is an uncertain process in which events, in uncertain times, occur continuously and independently of each another (Liu, 2012). In addition, a canonical process is defined as a stationary independent increment uncertain process whose increments are known as normal uncertain variables which follow a normal uncertainty distribution (Liu, 2012). Uncertain calculus was pioneered by Liu (2009). A canonical process is a foundation of uncertain calculus. The aim of uncertain calculus is to deal with differentiation and integration of uncertain processes. For more information, the reader is referred to, among other sources, Liu (2008, 2009, 2012, 2013), Wang and Peng (2014), You (2009), Gao (2009), and Liu and Ha (2010).

One of the prominent examples of portfolio insurance techniques is the CPPI. The notion of CPPI was first made known by Perold (1986) for fixed income instruments and Black and Jones (1987) for equity instruments.

## CPPI dynamics

Dynamic asset allocation (DAA) techniques refer to portfolio management approaches that focus on shifting portfolios between risky and risk-free asset classes in response to investor demands and market developments (Trippi and Harriff, 1991). The shifting of portfolios in DAA techniques is done throughout the investment period. Portfolio insurance is one area of DAA techniques which is rising in popularity.

The CPPI techniques are widely commended for their principal protection feature, simplicity and flexibility. Adopting CPPI strategies allows an investor to limit the portfolio downside risk whilst maintaining the portfolio upside potential. Market participants who buy CPPI participate in the future returns of an investment through the shifting of financial resources between asset classes (Schied, 2014). When markets are rising an investor takes high risk by shifting the portfolio towards a risky asset. Onthe other hand, when markets are falling the market participant reduces the exposure in a risky asset by placing more financial resources into a risk-free asset. A high risky asset yields high returns as compared to a risk-free asset.

The aim of CPPI strategies is to guarantee the initial principal amount, $F$, at the end of the investment tenure, $T$. Generally, CPPI strategies are based on the cushion, $U_t$. The cushion is defined as the difference between the portfolio value, $V_t$, and the portfolio floor, $P_t$, that is, $U_t =_t -P_t$. Portfolio floor is a carefully calculated value at which the portfolio value is not permitted to fall below. In order to guarantee the initial principal investment at time $T$, a proportion of the portfolio is placed in a risky asset and the remainder is invested in a risky-free security. In this case, a market consists of two basic assets: a risky asset (stock) and a risk-free security (zero-coupon bond). The amount to be invested in a risky asset, $Y_t$, is determined by multiplying the participation rate, $m$, by the cushion, that is, $Y_t = mU_t$. A participation rate is the leverage ratio which indicates the amount of leverage the investor is prepared to take. The study assumes that $m \geq 1$ because of the principal protection characteristic of the CPPI approaches. In addition, the study assumes that $m$ is constant throughout the investment period [0, *T*]. After deducting $Y_t$ from $V_t$, the remainder, $A_t = V_t - Y_t$, is then placed in a risk-free asset. The following conditions hold,

(i). if $V_t > P_t$, a carefully determined amount of money is invested in a risky asset and the remainder is placed in a risk-free asset.

(ii). if $P_t \geq V_t$, all financial resources are placed in a risk-free asset to protect the initial principal investment.

(iii). at $t = 0$, $V_0 > P_0$.

To deal with the dynamics of the CPPI strategies, the study assumes that the underlying asset prices follow a classical Liu stock model with constant interest rate $r$ and constant volatility $\sigma$. The price process of the underlying asset, $S_t$, at time $t$ evolves according to a geometric canonical Liu process driven uncertain differential equation

$$dS_t = S_t[\mu dt + \sigma dC_t], \tag{1.1}$$

where $\mu$ and $\sigma$ are constants, and $C_t$ is a canonical Liu process.

On the other hand, the price process of a zero-coupon bond, $B_t$, at time $t$ is described by

$$dB_t = B_t r dt, \tag{1.2}$$

where $r$ is the risk-free interest rate.

From the dynamics of the CPPI strategy, the cushion satisfies the uncertain differential equation given by

$$\frac{dU_t}{U_t} = (m(\mu - r) + r)dt + m\sigma dC_t, \tag{1.3}$$

whose explicit solution is

$$U(t) = U(o)e^{(m(\mu - r)+r)t + m\sigma C_t}. \tag{1.4}$$

The conclusion that is derived from the dynamics of the cushion is that in the classical Liu stock model with continuous trading, the initial principal investment is always guaranteed. In the classical Liu stock model with continuous trading, CPPI strategies are not exposed to gap risk. Gap risk is the possibility that the portfolio value may fall below the lower bound during the tenure of the investment. When markets are falling the cushion assumes a minimum value of zero. As the cushion falls the amountto be invested in a risky asset also falls and vice versa. Shifting of financial resources between the asset classes provides an initial capital guarantee at time $T$.

The value of the insured portfolio is described by

$$V_t = P_t + (V_0 - P_t e^{-rt})e^{(m(\mu - r)+r)t + m\sigma C_t}. \tag{1.5}$$

Equation (1.5) also shows that, in continuous time, CPPI strategies always protect the initial principal investments. That is, assuming that stock price processes follow a classical Liu stock model, CPPI approaches are not exposed to gap risk. This isfeasible because a geometric canonical Liu process is a continuous path process. In this case, the investor pay-off at time $T$ is given by

$$max\{V_T, B_T\} \equiv B_T + max\{U_T, 0\} \tag{1.6}$$

The expected value of the insured portfolio is independent of $\sigma$ and is given by

$$E[V_t] = P_t + (V_0 - P_t e^{-rt})e^{(m(\mu - r)+r)t}. \tag{1.7}$$

A conclusion that is derived from formula (1.7) is that, if $\mu > r$, the expected return of the CPPI insured portfolio can be increased continuously, without any additional risk, by taking higher participation rates (Cont and Tankov, 2007 and Neftci, 2008).

The CPPI strategies guarantee the initial principal amount at the end of the investment period. In continuous time diffusion models such as a classical Liu stock model, CPPI approaches are not exposed to gap risk. That is, stock prices are not exposed to sudden, significant downward jumps.

However, in practice stock prices exhibit jumps and the belief degree of gap risk for CPPI techniques is non-zero. As a result, gap risk for CPPI approaches needs to be quantified. This study seeks to quantify gap risk, using the investment risk index, for CPPI strategies in uncertain markets when stock prices are described by jump-diffusion models. Jump-diffusion models give precise asset price processes. This research paper analyses the relationship between a participation rate and the value of the underlying insured portfolio. The significance of a participation rate in CPPI approaches is also examined. One main contribution of this research paper is the provision of a method for quantifying gap risk for CPPI strategies in uncertain markets.

The rest of the research paper is organised as follows: In section 2 the Preliminaries and Problem Formulation are examined; the Investment Risk Index is analysed in section 3 and the Conclusion is given in section 4.

## Preliminaries and Problem Formulation

Preliminaries and problem formulation are presented in this section. This study adopts uncertain risk analysis which is a tool used in uncertainty theory to quantify risk. To strengthen the relationship between uncertain risk analysis and uncertainty theory, Liu (2012) defines risk as the "accidental loss" plus "uncertain measure of such loss". Definitions of terms are considered first of all.

**Definition 2.1** (Liu, 2007) Suppose $\Gamma$ is a non-empty set and $\mathcal{L}$ is a $\sigma$-algebra over $\Gamma$. Every element $\Lambda$ in $\mathcal{L}$ is an event. An uncertain measure is defined as a set function $\mathcal{M}$: $\mathcal{L} \to [0,1]$ which satisfies the following four axioms:

- **Axiom 1:** (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$;
- **Axiom 2:** (Monotonicity Axiom) $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$ if $\Lambda_1 \subset \Lambda_2$;
- **Axiom 3:** (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
- **Axiom 4:** (Sub-additivity Axiom) For every countable sequence of events $\{\Lambda_1, \Lambda_2, ...\}$,

$\mathcal{M}\{\cup_i \Lambda_i\} \le \sum_i \mathcal{M}\{\Lambda_i\}$.

**Definition 2.2** (Liu, 2007) An uncertain variable is defined as a measurable function $\xi$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that for any Borel set $B$, $\{\xi \in B\}$ is an event.

**Definition 2.3** (Liu, 2013) Suppose $T$ is an index set and $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space. An uncertain process is defined as a measurable function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that at any time $t$, for any Borel set $B$, $\{X_t \in B\}$ is an event.

**Definition 2.4** (Liu, 2009) An uncertain process $C_t$ is a canonical Liu process if

- $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
- $C_t$ has stationary and independent increments,

- every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance $t^2$, whose uncertainty distribution is given by

$$\Phi(x) = (1 + exp(\frac{-\pi x}{\sqrt{3}t}))^{-1}, x \in \Re. \tag{2.1}$$

**Definition 2.5** (Liu, 2008) Suppose $\xi_1, \xi_2, ...$ are independent and identically distributed positive uncertain inter-arrival times. By definition, $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + ... + \xi_n$ for $n \geq 1$. Then the uncertain process

$$N(t) = \max_{n \geq 0}\{n \mid S_n \leq t\} \tag{2.2}$$

is called an uncertain renewal process.

**Definition 2.6** Filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, models the evolution of information or knowledge as time moves on (Haugh, 2010). As a result, filtration on ( $(\Gamma, \mathcal{L}, \mathcal{M})$ is an index set, $\{\mathcal{F}_t\}_{t\geq 0}$, of sub-$\sigma$-algebra of $\mathcal{F}$.

**Definition 2.7** (Liu, 2012) Suppose $V_t$ to be an uncertain process and consider $P_t$ to be a given level. Then the uncertain variable

$$\tau_{P_t} = \inf\{t \geq 0 \mid V_t = P_t\} \tag{2.3}$$

is the first hitting time that $V_t$ touches the level $P_t$.

From a mathematical point of view, the study considers an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and filtration, $\{\mathcal{F}_t\}_{t\geq 0}$, generated by a one dimensional canonical process, $\{C_t\}_{t\geq 0}$, specified in the model.

The study considers a market which consists of two basic assets: a risky asset (stock) and a zero-coupon bond (risky-free security). To present a computational framework of gap risk measures for CPPI strategies in uncertain markets, this research paper considers that the underlying asset price follows a Yu (2013) stock model.

The price process of a risky underlying asset, $S_t$, at time $t$ follows an uncertain differential equation with jumps given below

$$dS_t = S_t[\mu dt + \sigma dC_t + \lambda dN_t], \tag{2.4}$$

where $\mu$ is the stock drift, $\sigma$ is the stock diffusion and $\lambda$ is the stock renewal coefficient, $C_t$ is a canonical process and $N_t$ is a renewal process. A renewal process models jumps in uncertain processes.

Explicitly, the value of $S_t$ at time $t$ is given by

$$S_t = S_0 exp(\int_0^t \mu ds + \int_0^t \sigma dC_s)\prod_{i=1}^{N_t}(1+\lambda), \tag{2.5}$$

$$S_t = S_0(1+\lambda)^{N_t} e^{\mu t + \sigma C_t}. \tag{2.6}$$

The price process of a zero-coupon bond, $B_t$, at time $t$ evolves according to the equation

$$dB_t = B_t r dt, \tag{2.7}$$

where $r$ is the risk-free interest rate. The research paper assumes that $\mu > r$.

The CPPI strategies are self-financing. As a result, until $V_t$ reaches $P_t$ at time $\tau$ the portfolio value is described by

$$dV_t = (U_t + P_t - mU_t)\frac{dB_t}{B_t} + mU_t\frac{dS_t}{S_t}, \tag{2.8}$$

which can be simplified to

$$\frac{dU_t}{U_t} = \frac{dB_t}{B_t} - m\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}, \tag{2.9}$$

$$\frac{dU_t}{U_t} = (1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}. \tag{2.10}$$

Suppose $dG_t$ is an uncertain process which describes a relative change in the cushion, and is given by

$$dG_t = (1-m)\frac{dB_t}{B_t} + m\frac{dS_t}{S_t}, \tag{2.11}$$

equation (2.10) becomes

$$\frac{dU_t}{U_t} = dG_t. \tag{2.12}$$

The study borrows the concept of a discounted cushion, $U_t^* = \frac{U_t}{B_t}$, from Cont and Tankov (2007). To analyse gap risk for CPPI approaches, the pay-off at $t = 0$ is determined by discounting $B_T$ and $U_T$ by

$B_T$. The pay-offat $t=0$ is therefore described by

$$1+max\{\frac{U_T}{B_T},0\}\equiv 1+max\{U_T^*,0\} \tag{2.13}$$

Hence, by introducing the discounted cushion $U_t^*=\frac{U_t}{B_t}$, equation (2.12) becomes

$$U_t^*=U_0^*e^{G_t}. \tag{2.14}$$

After time $\tau$, all financial resources are invested in a zero-coupon bond. This indicates that, after time $\tau$ the discounted cushion remains unchanged. Therefore, the discounted CPPI insured portfolio value is given by

$$\frac{V_t}{B_t}=1+(\frac{V_0}{B_0}-1)e^{G_{t\wedge\tau}} \tag{2.15}$$

The exponential term can take negative numbers. As a result, the initial principal investment, $F$, is no longer guaranteed at time $T$.

A CPPI strategy exhibits a loss when $P_t\geq V_t$ for $t\in[0,T]$. Alternatively, a CPPI technique incurs a loss when $U_t\leq 0$ or equivalently, $U_t^*\leq 0$. The price process of a zero-coupon bond is considered to be continuous. As aresult, $\{\tau\leq T\}$, $\{U_T\leq 0\}$ and $\{U_T^*\leq 0\}$ indicate the same thing. The above three events show that gap risk for CPPI strategies is encountered during the investment tenure $[0,T]$.

The main aim of this study is to analytically quantify gap risk for CPPI strategies in uncertain markets using the investment risk index. Investors' problem is to analyse the relationship between the participation rate and the value of the insured portfolio. The investors want to examine the significance of the participation rate in CPPI approaches. Portfolio value volatility depends on $m$. Therefore, in practice, $m$ should be gazetted in relation to several gap risk measures over a certain time period.

## Gap Risk Analysis

This paper analyses one gap risk metric which is the investment risk index.

### Stock Price Shocks and Gap Risk

It is widely understood and documented that in practice CPPI techniques are exposed to gap risk which emanates from sudden, significant downside jumps in portfolio value (Cont and Tankov, 2007). Stock prices are not continuous because of uncertainties such as time constraints, liquidity challenges and wars (Yu, 2013). In this research paper, the gap risk for CPPI techniques is considered to originate from sudden downward jumps in stock prices.

In the real world, limits on short sales and borrowing negatively affect the participation rate. A participation rate also depends on the risk tolerance of the investor. As a result, it is practically impossible to continuously increase the portfolio expected return by taking higher participation rates without additional risk.

In the presence of jumps in stock price, the cushion satisfies the uncertain differential equation described by

$$dU_t = (m(\mu - r) + r)U_t dt + m\sigma U_t dC_t + m\lambda U_t dN_t, \tag{3.1}$$

whose explicit solution is given by

$$U_t = U_0 exp(\int_0^t (m(\mu - r) + r)ds + \int_0^t m\sigma dC_s \prod_{i=1}^{N_t} (1 + m\lambda), \tag{3.2}$$

$$U_t = U_0 (1 + m\lambda)^{N_t} e^{(m(\mu - r) + r)t + m\sigma C_t}. \tag{3.3}$$

Equation (3.1) shows that as the cushion, $U_t$, takes very small values at time $t$, the first and second terms on the right side of the equation would be non-negative. On the other hand, as $U_t$ approaches zero at time $t$, a sudden downward underlying asset price jump causes the third term on the right side of equation (3.1) to take negative values. The dynamics of the cushion indicates that $dU_t$ can take negative values. Hence, the initial principal investment is not always protected.

Since $U_t = V_t - P_t$, in the presence of jumps in stock prices, the portfolio value evolves according to the following equation

$$V_t = P_t + (V_0 - P_0 e^{-rt})(1 + m\lambda)^{N_0} e^{(m(\mu - r) + r)t + m\sigma C_t}. \tag{3.4}$$

As a result, the expected CPPI insured portfolio value is given by

$$E[V_t] = P_t + (V_0 - P_t e^{-rt})(1 + m\lambda)^{N_t} e^{(m(\mu - r) + r)t}. \tag{3.5}$$

With reference to equation (3.5), it is practically impossible to indefinitely increase the CPPI insured portfolio return through taking higher participation rates without increasing the underlying risk. A participation rate which is greater than one, $m > 1$ magnifies jumps in the underlying risky asset price. The higher the participation rate in a bullish market, the higher the rate at which portfolio value increases. On the other hand, the higher the participation rate in a bearish market, the higher the rate at which portfolio value approaches the lower bound. This indicates that the CPPI insured portfolio value is positively correlated with the participation rate.

**Loss Function**

A CPPI strategy is a system with factors $V_t$ and $P_t$. Some specified loss in a CPPI technique depends on $V_t$ and $P_t$. Generally, the CPPI insured portfolio exhibits a loss if $P_t \geq V_t$. The study uses a loss function to represent such a loss. A loss function is the foundation of several gap risk measures.

**Definition 4.1** (Liu, 2012) If a system has factors $\xi_1, \xi_2, ..., \xi_n$, a function $f$ is called a loss function if some stated loss occurs if and only if

$$f(\xi_1, \xi_2, ..., \xi_n) > 0. \tag{3.6}$$

Consequently, a loss in a CPPI approach occurs if and only if

$$f(V_t, P_t) \geq 0. \tag{3.7}$$

Therefore, a CPPI strategy's loss function is described by

$$f(V_t, P_t) = -(V_0 - P_t e^{-rt})(1 + m\lambda)^{N_t} e^{(m(\mu-r)+r)t+m\sigma C_t} \geq 0. \tag{3.8}$$

**Investment Risk Index**

The investment risk index is important when a portfolio value is not allowed to fall below a certain benchmark during the tenure of the investment.

**Definition 4.2** (Liu, 2012) Suppose an investor has a number of investments, $n$, with returns that are uncertain variables $\xi_1, \xi_2, ..., \xi_n$. If a loss is described by $\xi_1 + \xi_2 + ... + \xi_n < c$, where $c$ is a pre-specified benchmark, then the investment risk index is given by

$$Risk = \mathcal{M}\{\xi_1 + \xi_2 + ... + \xi_n < c\}. \tag{3.9}$$

From the definition above, the investment risk index for the CPPI techniques is given by

$$Risk = \mathcal{M}\{V_t \leq P_t\} \tag{3.10}$$

The portfolio value falls down below the floor if the underlying risky security price falls down by more than $\frac{1}{m}$ during the investment period.

**Proposition.** Suppose $dG(t)$ can be rewritten as

$$dG_t = dG_t^c + dG_t^n, \tag{3.11}$$

where $dG_t^c$ refers to a continuous process and $dG_t^n$ indicates a renewal process with a jump measure $\lambda$.

Uncertain renewal process inter-arrival times are independently and identically distributed (i.i.d.). The sequence of inter-arrival times of an uncertain renewal process are exponentially distributed. Therefore, the investment risk index is given by

$$\mathcal{M}[\exists t \in [0,T] : V_t \le P_t] = 1 - e^{(-T\int_{-\frac{1}{m}}^{\infty} \lambda(dx))}. \tag{3.12}$$

**Proof.** An uncertain renewal process, $\{N_t, t \ge 0\}$, with a jump measure, $\lambda$, has an exponential inter-occurrence distribution. The exponential distribution models the time until the occurrence of a particular thing. As a result,

$$\mathcal{M}\{V_t \le P_t\} = 1 - e^{-\lambda x}. \tag{3.13}$$

The number of jumps of a renewal process with a jump measure $\lambda$, over a time period $(c,b]$, whose sizes fall in $(-\infty, -\frac{1}{m}]$ is an uncertain random variable with intensity $(b-c)\lambda((-\infty, -\frac{1}{m}])$.

**Corollary.**

Alternatively,

$$\mathcal{M}\{V_t > P_t\} = e^{-\lambda x}. \tag{3.14}$$

That is, whenever the CPPI insured portfolio stays above the floor, the investment risk index is given by

$$\mathcal{M}[\exists t \in [0,T] : V_t > P_t] = e^{(-T\int_{-\frac{1}{m}}^{\infty} \lambda(dx))}. \tag{3.15}$$

**Remark.** The investment risk index equals zero if the stock price process or its driving process is continuous.

# Conclusion

Currently, portfolio insurance is based on stochastic finance theory but the emergence of uncertainty theory has shifted some attention towards uncertain finance theory. This research paper analyses portfolio insurance in uncertain markets. The general understanding is that CPPI techniques are not exposed to gap risk. However, in reality, it is not correct to say that CPPI approaches are gap risk free. Gap risk of CPPI strategies emanates from sudden, significant downward jumps in the underlying risky asset prices. As a result, gap risk for CPPI strategies needs to be quantified, priced and hedged. The study proposes the investment risk index as one of the risk metrics for quantifying gap risk. As an extension of this peace of work, various risk measures such as expected loss, value-at-risk and tail value-at-risk are going to be examined in the forthcoming work.

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